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Abstract

We investigate stochastic homogenization for some degenerate quasilinear parabolic PDEs. The underlying nonlinear operator degenerates along the space variable, uniformly in the nonlinear term: the degeneracy points correspond to the degeneracy points of a reference diffusion operator on the random medium.

Assuming that this reference diffusion operator is ergodic, we can prove the homogenization property for the quasilinear PDEs, by means of the first order approximation method. The (nonlinear) limit operator needn't be nondegenerate. Concrete examples are provided.

Keywords : Stochastic homogenization; parabolic PDE; nonlinear PDE; degenerate PDE; first order approximation; ergodic operator.

1 Introduction

We are interested in the asymptotic behavior, as the parameter ε vanishes, of the solutions on $]0,T] \times \mathcal{O}$ of the PDEs

$$(1) \qquad \partial_t u^\varepsilon(t,x) - \operatorname{div}\big(a(\omega, x/\varepsilon, \nabla_x u^\varepsilon(t,x))\big) + f\left(\omega, x/\varepsilon, x, u^\varepsilon(t,x), \nabla_x u^\varepsilon(t,x)\right) = 0,$$

$(t,x) \in]0,T] \times \mathcal{O}$, with $u^\varepsilon(0,x) = 0$ for $x \in \mathcal{O}$ and $u^\varepsilon(t,x) = 0$ for $(t,x) \in]0,T] \times \partial\mathcal{O}$. In these equations, T denotes a positive real and $\mathcal{O} \subset \mathbb{R}^d$ a bounded convex open set of class $\mathcal{C}^{2+\alpha}$, for $\alpha > 0$. The parameter ω evolves in a random medium Ω, endowed with a σ-algebra $\mathcal{G}$ and a probability measure μ, with suitable stationarity and ergodicity properties. For all $x' \in \mathbb{R}^d$, $y \in \mathbb{R}$ and $z \in \mathbb{R}^d$, the fields $(\omega, x) \mapsto a(\omega, x, z)$ and $(\omega, x) \mapsto f(\omega, x, x', y, z)$ are stationary.

The main interest of our work lies in the possible degeneracies of the leading elliptic parts $-\mathrm{div}(a(\omega, x/\varepsilon, \nabla_x u^\varepsilon(t,x)))$, $\varepsilon > 0$. Both in the periodic and stochastic cases, the underlying field a is, in many papers devoted to homogenization, assumed to be strictly monotone with respect to the gradient term (i.e. uniformly elliptic in the linear framework). For example, in the recent work by Efendiev and Pankov [3] devoted to time-space homogenization of nonlinear random parabolic operators, the leading part fulfills a nondegeneracy property. However, as pointed out in earlier papers, the uniform ellipticity condition is far from being minimal. For example, in a series of articles, De Arcangelis and Serra Cassano [2], Paronetto and Serra Cassano [17] and Paronetto [18, 19] investigate the periodic homogenization of a class of degenerate linear equations. Loosely speaking, the diffusion coefficient is controlled by the identity matrix up to a scalar function that satisfies a Muckenhoupt condition. In a similar spirit, Huang et al. [8] consider nonlinear equations with periodic coefficients and Engström et al. [4] investigate homogenization of nonlinear random operators.

Our work relies on a different observation, which permits to deal with more general degeneracies (see Section 4 for a detailed comparison). In the linear case, the ellipticity assumption can be replaced by an ergodicity assumption on the underlying nonrescaled operator (see e.g. Rhodes [20, 21]). Indeed, if a has the form $a(\omega, x, z) = \hat{a}(\omega, x) z$ for a symmetric matricial field $\hat{a}(\omega, \cdot)$ with entries in $H^1_{\mathrm{loc}}(\mathbb{R}^d)$, the leading part $-\mathrm{div}(\hat{a}(\omega, x)\nabla_x \cdot)$ induces a self-adjoint operator on the random medium Ω, denoted by $\boldsymbol{L}^{\hat{a}}$ (see e.g. Papanicolaou and Varadhan [15, 16]). If this operator is ergodic, that is if the invariant functions for the associated semigroup are the constant functions, then the homogenization property holds for the rescaled operators. To adapt this idea to the nonlinear case, we assume that a in (1) has the form

$$a(\omega, x, z) = \tilde{\sigma}(\omega, x) A(\omega, x, \tilde{\sigma}(\omega, x) z), \ \omega \in \Omega, \ x, z \in \mathbb{R}^d, \tag{2}$$

where $A(\omega, x, z)$ is a strictly monotone vector with respect to z, uniformly in (ω, x), and $\tilde{\sigma}(\omega, x)$ is a symmetric matrix such that $\tilde{a}(\omega, \cdot)$, with $\tilde{a}(\omega, x) = \tilde{\sigma}(\omega, x)\tilde{\sigma}(\omega, x)$, has entries in $H^1_{\mathrm{loc}}(\mathbb{R}^d)$ and the operator $\boldsymbol{L}^{\tilde{a}}$ is ergodic on the random medium Ω, as explained above. This factorized form for the diffusion coefficient explains the title of our work: in (1), the rescaled operators degenerate along the space variable, uniformly with respect to the gradient term. We make a similar assumption on the source term f and assume that it may be expressed as $f(\omega, x, x', y, z) = F(\omega, x, x', y, \tilde{\sigma}(\omega, x) z)$, $\omega \in \Omega$, $x, x' \in \mathbb{R}^d$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, where F is Lipschitz continuous in y and z, uniformly in ω, x and x'.

We are then able, see Theorem 3.6, to establish the homogenization property for the solutions $(u^\varepsilon)_{\varepsilon>0}$: we prove that they converge in $L^\infty([0,T],L^2(\Omega\times\mathcal{O}))$ towards the solution of a limit equation, whose form is detailed below. We also manage to describe, in a strong sense, the asymptotic behavior of the gradients of $(u^\varepsilon)_{\varepsilon>0}$ and prove, in particular, their convergence up to a corrector term. We emphasize that our convergence result is an annealed version of the homogenization property unlike [3] where the convergence is stated for almost every realization of the stationary field. Even in the linear case, it seems that it is the price to pay for considering possibly degenerate operators.

The key point in our proof is a nonlinear version of the Birkhoff ergodic theorem on the random medium for quantities of the form

$$\int_0^T\int_{\mathcal{O}} h(\omega,x/\varepsilon,x,u^\varepsilon(t,x))dx\,dt. \tag{3}$$

The word *nonlinear* indicates that the functionals that we investigate depend on the solutions $(u^\varepsilon)_{\varepsilon>0}$. Using the ergodicity of $\boldsymbol{L}^{\tilde a}$, we manage to prove an averaging property for (3) with respect to the highly oscillating variable. Loosely speaking, under suitable conditions on h, (3) is close, in $L^1(\Omega\times[0,T]\times\mathcal{O})$, to $\int_\Omega\int_0^T\int_{\mathcal{O}} h(\omega,0,x,u^\varepsilon(t,x))dxdtd\mu(\omega)$.

Of course, the ergodicity of $\boldsymbol{L}^{\tilde a}$ is deeply connected to the geometry of the degeneracies of the matrix $\tilde a$: the coefficients $\tilde a$ and a are allowed to degenerate in certain directions only or to vanish on sets of null Lebesgue measure only. To the best of our knowledge, the only paper where the coefficient diffusion may vanish on sets of non-zero Lebesgue measure is due to Hairer and Pardoux [7]. In this work, the medium is periodic so that the authors can consider nondivergence operators with a non-zero drift. The role of the drift is crucial: it permits to preserve the ergodicity property on the areas where the diffusion coefficient vanishes. However, to make up for these local strong degeneracies, the authors require the existence of a strongly regularizing open area, so that the underlying diffusion coefficients cannot degenerate on the whole space. On the opposite, this situation is allowed in our setting (see Subsection 9.3 in Rhodes [21]).

In our framework, there are two main technical difficulties: first, the random medium is not compact and specific arguments to the periodic case fall short; second, for $\varepsilon>0$, the solution u^ε to (1) belong, at time t, to a subspace, denoted by $H_0^{\tilde\sigma(\omega,\cdot/\varepsilon),1}(\mathcal{O})$, of $L^2(\mathcal{O})$ which is larger than $H_0^1(\mathcal{O})$ and for which the classical $H_0^1(\mathcal{O})\hookrightarrow L^2(\mathcal{O})$ compactness arguments fail (loosely speaking, $H_0^{\tilde\sigma(\omega,\cdot/\varepsilon),1}(\mathcal{O})$ is the space of functions $\varphi\in L^2(\mathcal{O})$ such that $\tilde\sigma(\omega,\cdot/\varepsilon)\nabla\varphi$ exists in $L^2(\mathcal{O})$). For this reason, the G convergence theory (see Efendiev and Pankov [3], Pankov [13] and Svanstedt [22]) or refinements of this method (see the previous cited articles [2, 17, 18, 19] for Sobolev embeddings of suitable weighted spaces) fall short of establishing the homogenization property.

We thus use the first order approximation method (see Jikov, Kozlov and Oleinik [9], Chapter 7), that is we seek an approximation of u^ε of the form

$$u^\varepsilon(t,x) \sim \bar{u}(t,x) + \chi^{\varepsilon^2}\big(\omega, x/\varepsilon, \nabla_x \bar{u}(t,x)\big), \tag{4}$$

where $\chi^\lambda(\omega,x,z)$ denotes, for every $z \in \mathbb{R}^d$, an approximate corrector, that is a stationary field, solution of the equation

$$\lambda\chi^\lambda(\omega,x,z) - \operatorname{div}\big(\tilde{\sigma}(\omega,x)A(\omega,x,\tilde{\sigma}(\omega,x)z + \nabla_x\chi^\lambda(\omega,x,z))\big) = 0,$$

and $\bar{u}$ stands for the solution of the presumed limit equation.

The main difficulty in (4) is that $\bar{u}$ and χ^λ are not differentiable in all the directions of the space because of the degeneracies of a. In short, the field $\chi^\lambda(\omega,x/\varepsilon,z)$ is just differentiable along the matrix $\tilde{\sigma}(\omega,x/\varepsilon)$ (i.e. $\nabla_x\chi^\lambda(\omega,x/\varepsilon,z)$ doesn't exist but we can give a sense to $\tilde{\sigma}(\omega,x/\varepsilon)\nabla_x\chi^\lambda(\omega,x/\varepsilon,z)$). Similarly, the solution $\bar{u}$ is just differentiable along the matrix ς, equal to the square root of the effective diffusion coefficient α associated to the reference matrix $\tilde{a}$, i.e. $\varsigma = \alpha^{1/2}$. As a consequence, we have to develop a tedious regularization procedure to overcome the lack of differentiability in (4) (see Section 6 in the paper).

The reason why $\bar{u}$ is not differentiable in all the directions of the space is simple: the limit equation may be degenerate. At this step, we mention that this situation doesn't happen under the Muckenhoupt condition introduced in [2, 4, 8, 17, 18, 19] (see Section 4 for a detailed discussion). In our setting, the limit equation has the form

$$\partial_t\bar{u}(t,x) - \operatorname{div}\big(\bar{A}(\nabla_x\bar{u}(t,x))\big) + \hat{F}\big(x,\bar{u}(t,x),\nabla_x\bar{u}(t,x)\big) = 0,\ (t,x) \in]0,T] \times \mathcal{O}, \tag{5}$$

with $\bar{u}(0,x) = 0$ for $x \in \mathcal{O}$ and $\bar{u}(t,x) = 0$ for $(t,x) \in]0,T] \times \partial\mathcal{O}$. We can show that the limit coefficient $\bar{A}$ can be factorized by ς, that is $\bar{A}(z) = \varsigma\hat{A}(\varsigma z),\ z \in \mathbb{R}^d$, for a strictly monotone vector $\hat{A}$. In particular, the equation (5) is degenerate if the rank of α is less than or equal to $d-1$. Similarly, the limit source term has the form $\bar{F}(x,y,z) = \hat{F}(x,y,\varsigma z),\ x \in \mathbb{R}^d,\ y \in \mathbb{R},\ z \in \mathbb{R}^d$, for a mapping $\hat{F}$, Lipschitz continuous with respect to (y,z), uniformly in x. To understand in a better way the geometry of the limit equation (5), we can think of the case where ς is diagonal. Up to a change of variables, this is always possible since ς has a diagonal form in a suitable orthonormal basis. In this case, the equation (5) may be seen as a system of nondegenerate nonlinear equations parameterized by the kernel of ς, or equivalently by the kernel of α. We then understand in a deeper way the regularity of the limit solution $\bar{u}$. Along the sections of the domain $\mathcal{O}$ with respect to the kernel of α, the regularity of $\bar{u}$ follows from classical PDE results. For example, if the initial coefficients a and f are smooth, we can prove that $\bar{u}$ is smooth along the image of α. The convexity of the domain plays a crucial role at this point: since $\mathcal{O}$ is smooth and convex, the sections of the domain are regular.

The last question the reader may ask is the following: what can be said about the rank of α ? To be honest, this is a difficult question. We refer to Hairer and Pardoux [7] for a general discussion on this question in a different framework than ours. In our specific setting, we just provide two interesting examples: we first expose a surprising situation where the homogenized coefficient degenerates (and may even reduce to zero) in spite of strong nondegeneracy conditions of the initial coefficient over a domain with full Lebesgue measure; in the second example, we show that α may be uniformly elliptic even if $\tilde{a}$ is not (see Section 4 in the paper).

We now present the organization of the paper. In Section 2, we describe the random medium and expose the different assumptions. In Section 3, we detail the main results of the paper. In Section 4, we provide several examples. In Section 5, we investigate the corrector equations and discuss a nonlinear version of the ergodic theorem. The proof of the homogenization property if detailed in Section 6 and the geometry of the limit equation is discussed in Section 7.

2 Setup and assumptions

Random medium. Following [9], we introduce the following

Definition 2.1. *Let $(\Omega, \mathcal{G}, \mu)$ be a probability space and $\left\{\tau_x; x \in \mathbb{R}^d\right\}$ a group of measure preserving transformations acting ergodically on Ω:*

1) $\forall A \in \mathcal{G}, \forall x \in \mathbb{R}^d$, $\mu(\tau_x A) = \mu(A)$,

2) If for any $x \in \mathbb{R}^d$ $\tau_x A = A$, then $\mu(A) = 0$ or 1,

3) For any measurable function $\boldsymbol{g}$ on $(\Omega, \mathcal{G}, \mu)$, the function $(x, \omega) \mapsto \boldsymbol{g}(\tau_x \omega)$ is measurable on $(\mathbb{R}^d \times \Omega, \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{G})$.

The expectation with respect to the random medium is denoted by $\mathbb{E}$. In what follows we use the bold type to denote a function $\boldsymbol{g}$ from Ω into $\mathbb{R}$ (or more generally into $\mathbb{R}^n$, $n \geq 1$) and the unbold type $g(\omega, x)$ (or just $g(x)$ when possible) to denote the associated representation mapping $(\omega, x) \mapsto \boldsymbol{g}(\tau_x \omega)$. Similarly, for a family $(\boldsymbol{g}(\cdot, z))_{z \in \mathbb{R}^n}$, $n \geq 1$, of functions from Ω into $\mathbb{R}^n$, we denote by $g(\omega, x, z)$ (or just $g(x, z)$ when possible) the mapping $(\omega, x, z) \mapsto \boldsymbol{g}(\tau_x \omega, z)$. The space of square integrable functions on $(\Omega, \mathcal{G}, \mu)$ is denoted by $L^2(\Omega)$, the usual norm by $|\cdot|_2^{\Omega}$ and the corresponding inner product by $(\cdot\,, \cdot)_2^{\Omega}$. Then, the operators on $L^2(\Omega)$ defined by $T_x \boldsymbol{g}(\omega) = \boldsymbol{g}(\tau_x \omega)$ form a strongly continuous group of unitary maps in $L^2(\Omega)$. The group possesses d generators defined by $D_i \boldsymbol{g} = \lim_{h \to 0} h^{-1}[T_{he_i} \boldsymbol{g} - \boldsymbol{g}]$ if exists, which are closed and densely defined. Setting $\mathcal{C} =$

Span $\{\boldsymbol{g} \star \varphi; \boldsymbol{g} \in L^\infty(\Omega), \varphi \in C_c^\infty(\mathbb{R}^d)\}$, with $\boldsymbol{g} \star \varphi(\omega) = \int_{\mathbb{R}^d} \boldsymbol{g}(\tau_x\omega)\varphi(x)\,dx$, the space $\mathcal{C}$ is dense in $L^2(\Omega)$ and $\mathcal{C} \subset \mathrm{Dom}(D_i)$ for all $1 \le i \le d$, with $D_i(\boldsymbol{g} \star \varphi) = -\boldsymbol{g} \star \partial\varphi/\partial x_i$. If $\boldsymbol{g} \in \mathrm{Dom}(D_i)$, we also have $D_i(\boldsymbol{g} \star \varphi) = D_i\boldsymbol{g} \star \varphi$.

Structure of the PDE. As explained in Introduction, we assume in the whole paper that the nonlinearities of order one can be factorized by a reference matrix. We thus introduce the following

Definition 2.2. *Given a function φ from $\mathbb{R} \times \mathbb{R}^d$ into $\mathbb{R}$ (the definition below may be adapted in a trivial way when the values of φ belong to a normed space), a $d \times d$ symmetric matrix σ and a positive constant C, φ is said $((1,\sigma), C)$-Lipschitz continuous if for all $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^d$*

$$|\varphi(y,z) - \varphi(y',z')| \le C\big(|y-y'| + |\sigma(z-z')|\big). \tag{6}$$

Given a function φ from $\mathbb{R}^d$ into $\mathbb{R}^d$, a $d \times d$ symmetric matrix σ and a constant $C \ge 1$, φ is said (σ, C)-strictly monotone if for all $\zeta \in \mathbb{R}^d$ and for all $z, z' \in \mathbb{R}^d$

$$\begin{array}{lrll} \text{(i)} & & \langle \varphi(0), \zeta \rangle & \le C\,|\sigma\zeta| \\ \text{(ii)} & & \langle \varphi(z) - \varphi(z'), \zeta \rangle & \le C\,|\sigma(z-z')|\,|\sigma\zeta| \\ \text{(iii)} & C^{-1}|\sigma(z-z')|^2 \le & \langle \varphi(z) - \varphi(z'), z - z' \rangle & \end{array} . \tag{7}$$

We can prove, for a $((1,\sigma), C)$-Lipschitz continuous function φ, that there exists a $((1, I_d), C)$-Lipschitz continuous function Φ (I_d denotes the identity matrix of size d) such that $\varphi(y,z) = \Phi(y, \sigma z)$ ([1]). Similarly, for a (σ, C)-strictly monotone function φ, there exists an (I_d, C)-strictly monotone function Φ such that $\varphi(z) = \sigma\Phi(\sigma z)$ ([2]) . In both cases, the function Φ can be constructed with the same regularity as the function φ.

From now on, the coefficients $\boldsymbol{a} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ and $\boldsymbol{f} : \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ denote measurable functions with respect to the underlying product σ-fields. In the whole paper, we assume that there exists a constant $\Lambda \ge 1$ such that

Assumption 2.3 (Control). *There exists a measurable function $\tilde{\boldsymbol{\sigma}} : \Omega \to \mathcal{S}_d(\mathbb{R})$ (set of $d \times d$ symmetric real matrices), bounded by Λ, such that $\boldsymbol{a}(\omega, \cdot)$ is $(\tilde{\boldsymbol{\sigma}}(\omega), \Lambda)$-strictly monotone for each $\omega \in \Omega$.*

[1] The matrix σ may be expressed as $\sigma = M\mathrm{diag}[\lambda_1, \ldots, \lambda_r, 0, \ldots]M^*$, for r reals $\lambda_1, \ldots, \lambda_r$, different from zero, and for an orthogonal matrix M ($\mathrm{diag}[\lambda_1, \ldots, \lambda_r, 0, \ldots]$ stands for the diagonal matrix of size d whose diagonal elements are equal to $\lambda_1, \ldots, \lambda_r, 0, \ldots$). We set $\Phi(y,z) = \varphi(y, \sigma^{-1}z)$ with $\sigma^{-1} = M\mathrm{diag}[1/\lambda_1, \ldots, 1/\lambda_r, 0, \ldots]M^*$.

[2] Same construction as above, but with $\Phi(z) = \sigma^{-1}\varphi(\sigma^{-1}z) + C^{-1}M\mathrm{diag}[0, \ldots, 1\ldots]M^*z$ (the number of "0" is r and the number of "1" is $d - r$).

Assumption 2.4 (**Regularity**). *For each fixed $\omega \in \Omega$ and for all $x \in \mathcal{O}$, $|\boldsymbol{f}(\omega, x, 0, 0)| \leq \Lambda$ and $\boldsymbol{f}(\omega, x, \cdot, \cdot)$ is $((1, \tilde{\boldsymbol{\sigma}}(\omega)), \Lambda)$-Lipschitz continuous.*

Assumption 2.5 (**Ergodicity**). *The entries of the mapping $\omega \in \Omega \mapsto \tilde{\boldsymbol{a}}(\omega) = \tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\sigma}}(\omega)$ belong to $\cap_{1 \leq i \leq d} D_i$. In particular, we can define the operator $\tilde{\boldsymbol{S}}(\cdot) = (1/2)\sum_{i,j=1}^{d} D_i(\tilde{\boldsymbol{a}}_{i,j} D_j \cdot)$ on $\mathcal{C}$ and consider its Friedrichs extension (see [5, p. 53]), which is self-adjoint. We then assume that the semi-group generated by $\tilde{\boldsymbol{S}}$ is ergodic, that is, its invariant functions are μ almost surely constant (see e.g. Rhodes [20]).*

From Assumption 2.3, we can express $\boldsymbol{a}$ as $\boldsymbol{a}(\omega, z) = \tilde{\boldsymbol{\sigma}}(\omega)\boldsymbol{A}(\omega, \tilde{\boldsymbol{\sigma}}(\omega)z)$, for an (I_d, Λ)-strictly monotone coefficient $\boldsymbol{A}(\omega, \zeta)$. We can choose a jointly measurable version of $\boldsymbol{A}$. Similarly, we can write $\boldsymbol{f}(\omega, x, y, z) = \boldsymbol{F}(\omega, x, y, \tilde{\boldsymbol{\sigma}}(\omega)z)$, where $\boldsymbol{F}$ is a measurable mapping such that $\boldsymbol{F}(\omega, x, \cdot, \cdot)$ is $((1, I_d), \Lambda)$-Lipschitz continuous for all $\omega \in \Omega$ and $x \in \mathbb{R}^d$.

Notation. We put $\mathcal{Q}_T = [0, T] \times \mathcal{O}$. For a measurable function h defined on a measurable space $(E, \mathcal{T})$ endowed with a finite measure π, we denote by $\|h\|_2^E$ and $\|h\|_\infty^E$ the L^2 and L^∞ norms of h on this space. For a function $h \in L^1(\Omega \times \mathcal{O})$, the notation $\mathbb{M}^{dx}[\boldsymbol{h}(\cdot, x)]$ stands for $\mathbb{E}\int_{\mathcal{O}} \boldsymbol{h}(\omega, x)dx$. For a function $h \in L^1(\Omega \times \mathcal{Q}_T)$, $\mathbb{M}^{dt,dx}[\boldsymbol{h}(\cdot, t, x)]$ stands for $\mathbb{E}\int_0^T \int_{\mathcal{O}} \boldsymbol{h}(\omega, t, x)dtdx$ and, for $t \in [0, T]$, $\mathbb{M}_t^{ds,dx}[\boldsymbol{h}(\cdot, s, x)]$ stands for $\mathbb{E}\int_0^t \int_{\mathcal{O}} \boldsymbol{h}(\omega, s, x) dsdx$. Similarly, for an element $A \in \mathcal{G} \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathcal{O})$, $\mathbb{Q}^{dt,dx}[A]$ stands for $\mathbb{M}^{dt,dx}[\mathbf{1}_A]$.

3 Main Results

3.1 Solvability of the PDEs

For a given bounded function $\Psi : \mathcal{O} \to \mathbb{R}^{d \times d}$, for which the entries of $\Psi\Psi^*$ are in $H^1(\mathcal{O})$, we define $H_0^{\Psi,1}$ as the completion of $\mathcal{C}_K^\infty(\mathcal{O})$ (smooth functions on $\mathcal{O}$ with a compact support), with respect to the norm $N^\Psi : \varphi \in \mathcal{C}_K^\infty(\mathcal{O}) \mapsto [(\|\varphi\|_2^{\mathcal{O}})^2 + \|\Psi\nabla\varphi\|_2^{\mathcal{O}})^2]^{1/2}$.

By the regularity of Ψ, the quadratic form $\varphi \in \mathcal{C}_K^\infty(\mathcal{O}) \mapsto [N^\Psi(\varphi)]^2$ is closable, so that $H_0^{\Psi,1}$ may be seen as a subspace of $L^2(\mathcal{O})$. Equipped with the norm induced by N^Ψ, it is a Hilbert space. We put $H^{\Psi,-1} = (H_0^{\Psi,1})'$.

For $\omega \in \Omega$ and $\varepsilon > 0$, we put $V^{\varepsilon,\omega} = H_0^{\tilde{\sigma}(\omega,\cdot/\varepsilon),1}$. The closure of $\{\tilde{\sigma}(\omega, \cdot/\varepsilon)\nabla\varphi,\ \varphi \in \mathcal{C}_K^\infty(\mathcal{O})\}$ in $[L^2(\mathcal{O})]^d$ is denoted by $G^{\varepsilon,\omega}$. It is then clear that the mapping $\varphi \in \mathcal{C}_K^\infty(\mathcal{O}) \mapsto \tilde{\sigma}(\omega, \cdot/\varepsilon)\nabla\varphi \in G^{\varepsilon,\omega}$ can be extended to the whole $V^{\varepsilon,\omega}$, so that a function $g \in V^{\varepsilon,\omega}$ admits a gradient along the direction $\tilde{\sigma}(\omega, \cdot/\varepsilon)$, denoted by $\nabla^{\tilde{\sigma}(\omega,\cdot/\varepsilon)}g$.

The mapping $\varphi \in V^{\varepsilon,\omega} \mapsto (\varphi, \nabla^{\tilde{\sigma}(\omega,\cdot/\varepsilon)}\varphi)$ is an isometry from $V^{\varepsilon,\omega}$ onto a closed subspace of $[L^2(\mathcal{O})]^{d+1}$. For this reason, $V^{\varepsilon,\omega}$ is separable.

Theorem 3.1. *For $\omega \in \Omega$ and $\varepsilon > 0$, there exists a unique function $u^{\varepsilon,\omega}$ in $L^2(]0,T[, V^{\varepsilon,\omega})$, with $\partial_t u^{\varepsilon,\omega} \in L^2(]0,T[,(V^{\varepsilon,\omega})')$, satisfying for a.e. $t \in]0,T[$ and for all $\varphi \in V^{\varepsilon,\omega}$*

$$\int_{\mathcal{O}} \partial_t u^{\varepsilon,\omega}(t,x)\varphi(x)dx + \int_{\mathcal{O}} \big\langle A\big(\omega, x/\varepsilon, \nabla^{\tilde{\sigma}(\omega,\cdot/\varepsilon)} u^{\varepsilon,\omega}(t,x)\big), \nabla^{\tilde{\sigma}(\omega,\cdot/\varepsilon)}\varphi(x)\big\rangle dx$$
$$+ \int_{\mathcal{O}} F\big(\omega, x/\varepsilon, x, u^{\varepsilon,\omega}(t,x), \nabla^{\tilde{\sigma}(\omega,\cdot/\varepsilon)} u^{\varepsilon,\omega}(t,x)\big)\varphi(x)dx = 0,$$

and verifying $u^{\varepsilon,\omega}(0,\cdot) = 0$. We say that the function $u^{\varepsilon,\omega}$ is the unique solution of (1).

Except in particular cases, the index ω will be omitted in $u^{\varepsilon,\omega}$.

Remark. Since $u^{\varepsilon,\omega} \in L^2(]0,T[, V^{\varepsilon,\omega})$ and $\partial_t u^{\varepsilon,\omega} \in L^2(]0,T[,(V^{\varepsilon,\omega})')$, we can prove that $u^{\varepsilon,\omega} \in \mathcal{C}([0,T], L^2(\mathcal{O}))$ (see [12, Th. 3.1 & Prop. 2.1, Ch. 1]). For this reason, the initial condition is well defined.

Proof. We can assume without loss of generality that $\varepsilon = 1$. We also assume for the moment that $\boldsymbol{f}(\omega, x, y, z)$ and $\boldsymbol{F}(\omega, x, y, z)$ don't depend on (y, z). We thus investigate the evolution problem

$$\text{(8)} \quad \partial_t u^1(t,x) - \operatorname{div}\big(\tilde{\sigma}(\omega)A(\omega, x, \nabla_x^{\tilde{\sigma}(\omega,\cdot)} u^1(t,x))\big) + f(\omega, x, x) = 0,\ (t,x) \in]0,T] \times \mathcal{O},$$

with the initial condition $u^{\varepsilon,\omega}(0,\cdot) = 0$. The nonlinear operator $\mathcal{A}^{1,\omega} : \varphi \in V^{1,\omega} \mapsto -\operatorname{div}\big(\tilde{\sigma}(\omega)A(\omega,\cdot,\nabla^{\tilde{\sigma}(\omega,\cdot)}\varphi(\cdot))\big) \in (V^{1,\omega})'$ is Lipschitz continuous and strictly monotone on $V^{1,\omega}$: this proves the unique solvability of the evolution equation when $\boldsymbol{f}$ and $\boldsymbol{F}$ don't depend on (y, z), see [11, Th. 1.2, Ch. 2]. The general case can be treated in a usual way, by means of the Picard fixed point theorem. □

Proposition 3.2. *For every $\omega \in \Omega$, we can find a version $\tilde{u}^{\varepsilon,\omega}$ of $u^{\varepsilon,\omega}$ such that the mapping $(\omega, t, x) \in \Omega \times [0,T] \times \mathcal{O} \mapsto (\tilde{u}^{\varepsilon,\omega}, \nabla^{\tilde{\sigma}(\omega,\cdot/\varepsilon)}\tilde{u}^{\varepsilon,\omega})(t,x)$ is jointly measurable with respect to the product σ-field $\mathcal{G} \otimes \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathcal{O})$. Without loss of generality, we can assume that, for all $(\omega, t) \in \Omega \times [0,T]$, $\tilde{u}^{\varepsilon,\omega}(t,\cdot)$ is a version of $u^{\varepsilon,\omega}(t,\cdot)$. In particular, $\tilde{u}^{\varepsilon,\omega} \in \mathcal{C}([0,T], L^2(\mathcal{O}))$. In what follows, we write $u^{\varepsilon,\omega}$ for $\tilde{u}^{\varepsilon,\omega}$.*

Proof. We can assume without loss of generality that $\varepsilon = 1$. We then follow the proof of Theorem 3.1. When the coefficients $\boldsymbol{f}(\omega, x, y, z)$ and $\boldsymbol{F}(\omega, x, y, z)$ don't depend on (y, z), the solvability of (8) follows from a Faedo-Galerkin approximation (see again [11, Sec. 1, Ch. 2]). The construction of an orthonormal basis of $V^{1,\omega}$ can be achieved in a measurable way with respect to the parameter ω: we first choose an orthonormal basis of $L^2(\mathcal{O})$ composed of smooth functions with a compact support and we then apply the Gram-Schmidt procedure. As a by-product, the Faedo-Galerkin approximations are jointly measurable. The limit, that is the solution of (8), admits a jointly measurable version, as written in the statement of Proposition 3.2.

Using a Picard iteration sequence, we can prove that the result remains true when $\boldsymbol{f}$ and $\boldsymbol{F}$ depend on (y,z). $\square$

The proof of the following estimate is left to the reader:

Proposition 3.3. *There exists a constant $C_{3.3}$ such that, for all $\varepsilon>0$, $\sup_{t\in[0,T]}\mathbb{M}^{dx}[|u^{\varepsilon}(t,x)|^2]+\mathbb{M}^{dt,dx}[|\nabla_x^{\tilde{\sigma}(x/\varepsilon)}u^{\varepsilon}(t,x)|^2]\leq C_{3.3}$.*

3.2 Limit Equation

We are now in position to introduce the homogenized problem (see (13)). As announced in Introduction, it may be degenerate. For this reason, we have to control the possible degeneracies with respect to a suitable norm on $\mathcal{C}_K^\infty(\mathcal{O})$, as done in Section 3.1 for the equation (1). This norm is induced by the effective diffusion coefficient associated to $\tilde{\boldsymbol{a}}$.

We prove in Section 5 the following

Proposition 3.4. *Define $\mathbb{D}$ as the closure in $(L^2(\Omega))^d$ of the set $\{\tilde{\boldsymbol{\sigma}}D\boldsymbol{\varphi},\ \boldsymbol{\varphi}\in\mathcal{C}\}$. Then, there exists an element $\tilde{\boldsymbol{\xi}}=(\tilde{\boldsymbol{\xi}}_1,\dots,\tilde{\boldsymbol{\xi}}_d)\in\mathbb{D}^d$ such that, for all $z\in\mathbb{R}^d$, $\boldsymbol{div}(\tilde{\boldsymbol{\sigma}}(\tilde{\boldsymbol{\sigma}}z+\tilde{\boldsymbol{\xi}}z))=0$ (the product $\tilde{\boldsymbol{\xi}}z$ stands for the combination $\sum_{i=1}^d z_i\tilde{\boldsymbol{\xi}}_i$: each function $\tilde{\boldsymbol{\xi}}_i$, $1\leq i\leq d$, is $\mathbb{R}^d$ valued so that $\tilde{\boldsymbol{\xi}}$ is a matricial function). It satisfies the following variational formula*

$$\inf_{\boldsymbol{\varphi}\in\mathbb{D}}\mathbb{E}\big[|\tilde{\boldsymbol{\sigma}}z+\boldsymbol{\varphi}|^2\big]=\mathbb{E}\big[|(\tilde{\boldsymbol{\sigma}}+\tilde{\boldsymbol{\xi}})z|^2\big]=\langle z,\alpha z\rangle,\tag{9}$$

with $\alpha=\mathbb{E}[(\tilde{\boldsymbol{\sigma}}+\tilde{\boldsymbol{\xi}})(\tilde{\boldsymbol{\sigma}}+\tilde{\boldsymbol{\xi}})^*]$. Moreover, for all $z\in\mathbb{R}^d$, the auxiliary problem

$$\boldsymbol{div}\big(\tilde{\boldsymbol{\sigma}}\boldsymbol{A}(\cdot,\tilde{\boldsymbol{\sigma}}z+\boldsymbol{\xi}(\cdot,z))\big)=0\tag{10}$$

admits a unique (weak) solution $\boldsymbol{\xi}(\cdot,z)$ in $\mathbb{D}$. Setting $\varsigma=\alpha^{1/2}$, the mapping $z\in\mathbb{R}^d\mapsto\tilde{\boldsymbol{\sigma}}z+\boldsymbol{\xi}(\cdot,z)\in\mathbb{D}$ is $(\varsigma,C_{3.4})$-Lipschitz continuous for some positive constant $C_{3.4}$:

$$\forall z,z'\in\mathbb{R}^d,\ \mathbb{E}[|\tilde{\boldsymbol{\sigma}}(z-z')+\boldsymbol{\xi}(\cdot,z)-\boldsymbol{\xi}(\cdot,z')|^2]\leq C_{3.4}|\varsigma(z-z')|^2.\tag{11}$$

Remark. By (11), we can find a jointly measurable version of $\boldsymbol{\xi}(\omega,z)$.

Theorem 3.5. *For $x\in\mathcal{O}$, $y\in\mathbb{R}$ and $z\in\mathbb{R}^d$, set*

$$\bar{A}(z)=\mathbb{E}\big[\tilde{\boldsymbol{\sigma}}\boldsymbol{A}(\cdot,\tilde{\boldsymbol{\sigma}}z+\boldsymbol{\xi}(\cdot,z))\big],\ \bar{F}(x,y,z)=\mathbb{E}\big[\boldsymbol{F}(\cdot,x,y,\tilde{\boldsymbol{\sigma}}z+\boldsymbol{\xi}(\cdot,z))\big].\tag{12}$$

Then, for some constant $C_{3.5} > 0$, $\bar{A}$ is $(\varsigma, C_{3.5})$-strictly monotone and, for all $x \in \mathcal{O}$, $|\bar{F}(x,0,0)| \leq C_{3.5}$ and $\bar{F}(x,\cdot,\cdot)$ is $((1,\varsigma), C_{3.5})$-Lipschitz continuous. As a consequence, the PDE defined on $]0,T] \times \mathcal{O}$ by

$$\partial_t \bar{u}(t,x) - \operatorname{div}\big(\bar{A}(\nabla_x \bar{u}(t,x))\big) + \bar{F}\big(x, \bar{u}(t,x), \nabla_x \bar{u}(t,x)\big) = 0, \ (t,x) \in]0,T] \times \mathcal{O}, \tag{13}$$

with a null boundary condition on $\{0\} \times \mathcal{O} \cup]0,T] \times \partial\mathcal{O}$, admits a unique solution $\bar{u} \in L^2(]0,T[, H_0^{\varsigma,1}(\mathcal{O}))$, with $\partial_t \bar{u} \in L^2(]0,T[, H^{\varsigma,-1}(\mathcal{O}))$.

Proof. We check the strict monotonicity of $\bar{A}$ (cf. (7)) by means of (9). Indeed, from Proposition 3.4, we know that, for all $z \in \mathbb{R}^d$, $\boldsymbol{A}(\cdot, \tilde{\boldsymbol{\sigma}} z + \boldsymbol{\xi}(\cdot, z))$ and $\tilde{\boldsymbol{\sigma}} z + \tilde{\boldsymbol{\xi}} z$ belong to $\mathbb{D}^{\perp}$. Hence, for $\zeta \in \mathbb{R}^d$,

$$\langle \bar{A}(0), \zeta \rangle = \mathbb{E}\big[\langle \tilde{\boldsymbol{\sigma}}\zeta, \boldsymbol{A}(\cdot, \boldsymbol{\xi}(\cdot,0))\rangle\big] = \mathbb{E}\big[\langle (\tilde{\boldsymbol{\sigma}} + \tilde{\boldsymbol{\xi}})\zeta, \boldsymbol{A}(\cdot, \boldsymbol{\xi}(\cdot,0))\rangle\big] \leq C\big(\mathbb{E}\big[|(\tilde{\boldsymbol{\sigma}} + \tilde{\boldsymbol{\xi}})\zeta|^2\big]\big)^{1/2}.$$

This proves (7.i). We turn to (7.iii). For $z, z', \zeta \in \mathbb{R}^d$, we claim

$$\begin{aligned} &\langle \bar{A}(z) - \bar{A}(z'), z - z' \rangle \\ &= \mathbb{E}\big[\langle \boldsymbol{A}(\cdot, \tilde{\boldsymbol{\sigma}} z + \boldsymbol{\xi}(\cdot,z)) - \boldsymbol{A}(\cdot, \tilde{\boldsymbol{\sigma}} z' + \boldsymbol{\xi}(\cdot,z')), \tilde{\boldsymbol{\sigma}}(z-z') + \boldsymbol{\xi}(\cdot,z) - \boldsymbol{\xi}(\cdot,z')\rangle\big] \\ &\geq \Lambda^{-1}\mathbb{E}\big[|\tilde{\boldsymbol{\sigma}}(z-z') + \boldsymbol{\xi}(\cdot,z) - \boldsymbol{\xi}(\cdot,z')|^2\big] \\ &\geq \Lambda^{-1} \inf_{\boldsymbol{\varphi} \in \mathbb{D}} \mathbb{E}\big[|\tilde{\boldsymbol{\sigma}}(z-z') + \boldsymbol{\varphi}|^2\big] = \Lambda^{-1}|\varsigma(z-z')|^2. \end{aligned} \tag{14}$$

We establish (7.ii). For $z, z', \zeta \in \mathbb{R}^d$, we claim

$$\begin{aligned} \langle \bar{A}(z) - \bar{A}(z'), \zeta \rangle &= \mathbb{E}\big[\langle \boldsymbol{A}(\cdot, \tilde{\boldsymbol{\sigma}} z + \boldsymbol{\xi}(\cdot,z)) - \boldsymbol{A}(\cdot, \tilde{\boldsymbol{\sigma}} z' + \boldsymbol{\xi}(\cdot,z')), \tilde{\boldsymbol{\sigma}}\zeta + \tilde{\boldsymbol{\xi}}\zeta\rangle\big] \\ &\leq \Lambda\big(\mathbb{E}\big[|\tilde{\boldsymbol{\sigma}}(z-z') + \boldsymbol{\xi}(\cdot,z) - \boldsymbol{\xi}(\cdot,z')|^2\big]\big)^{1/2}|\varsigma\zeta| \end{aligned} \tag{15}$$

Plugging (15) into (14), we deduce that $\mathbb{E}[|\tilde{\boldsymbol{\sigma}}(z-z') + \boldsymbol{\xi}(\cdot,z) - \boldsymbol{\xi}(\cdot,z')|^2] \leq \Lambda^2(\mathbb{E}[|\tilde{\boldsymbol{\sigma}}(z - z') + \boldsymbol{\xi}(\cdot,z) - \boldsymbol{\xi}(\cdot,z')|^2])^{1/2}|\varsigma(z-z')|$. By (15), we complete the proof of (7.ii). As a by-product, we deduce (11). We let the reader check the Lipschitz properties of $\bar{F}$ in y and z.

We investigate the solvability of (13) as in Theorem 3.1. □

3.3 Homogenization Property

We present below the homogenization property: the sequence $(u^{\varepsilon})_{\varepsilon>0}$ converges towards $\bar{u}$ in $L^{\infty}([0,T], L^2(\Omega \times \mathcal{O}))$ as ε tends to zero. We are also able to specify the convergence of the gradients: the distance in $L^2(]0,T[, L^2(\Omega \times \mathcal{O}))$ between $\nabla_x^{\tilde{\sigma}(x/\varepsilon)} u^{\varepsilon}$ and $\tilde{\sigma}(x/\varepsilon)\nabla_x \bar{u}(t,x) + \xi(x/\varepsilon, \nabla_x \bar{u}(t,x))$ tends to zero as ε vanishes.

The reader may object that $\tilde{\sigma}(x/\varepsilon)\nabla_x\bar{u}(t,x)$ and $\xi(x/\varepsilon,\nabla_x\bar{u}(t,x))$ are meaningless since the gradient of $\bar{u}$ doesn't exist in all the directions of the space. In fact, by means of standard convolution argument, we can find a sequence $(\varphi_n)_{n\geq 1}$ of measurable functions from $[0,T]\times\mathbb{R}^d$ into $\mathbb{R}$, such that $\varphi_n(t,\cdot)$ belongs to $\mathcal{C}^\infty(\mathcal{O})$ for every $t\in[0,T]$ and $(\varphi_n,\varsigma\nabla_x\varphi_n)\to(\bar{u},\nabla_x^{\varsigma}\bar{u})$ in $(L^2(]0,T[\times\mathcal{O}))^{d+1}$. By (11), the sequence $(\omega,t,x)\mapsto\tilde{\boldsymbol{\sigma}}(\omega)\nabla_x\varphi_n(t,x)+\boldsymbol{\xi}(\omega,\,\nabla_x\varphi_n(t,x))$ is a Cauchy sequence in $L^2(\Omega\times[0,T]\times\mathcal{O})$ and the limit doesn't depend on the choice of the approximating sequence $(\varphi_n)_{n\geq 1}$. It is denoted by $(\omega,t,x)\mapsto\tilde{\boldsymbol{\sigma}}(\omega)\nabla_x\bar{u}(t,x)+\boldsymbol{\xi}(\omega,\nabla_x\bar{u}(t,x))$.

Theorem 3.6. *Under Assumptions 2.3, 2.4 and 2.5,*

$$\lim_{\varepsilon\to 0}\Big\{\sup_{0\leq t\leq T}\mathbb{E}\int_{\mathcal{O}}\big|(u^\varepsilon-\bar{u})(t,x)\big|^2dx \\ +\mathbb{E}\int_0^T\int_{\mathcal{O}}\big|\nabla_x^{\tilde{\sigma}(x/\varepsilon)}u^\varepsilon(t,x)-[\tilde{\sigma}(x/\varepsilon)\nabla_x\bar{u}(t,x)+\xi(x/\varepsilon,\nabla_x\bar{u}(t,x))]\big|^2dt\,dx\Big\}=0.$$

4 Examples

4.1 An Example where the Effective Diffusion Matrix is Null

We now tackle the construction of a two-dimensional periodic example where the diffusion coefficient $\tilde{\boldsymbol{\sigma}}$ is uniformly elliptic over an open subset of $\mathbb{R}^2$ but the effective diffusion coefficient α is null. To this purpose, we define the 2π-periodic diffusion coefficient on $\mathbb{R}^2$ $\tilde{\boldsymbol{\sigma}}(x_1,x_2)=(1-\cos(x_1))(1-\cos(x_2))I_2$, where I_2 stands for the 2×2 identity matrix.

We first prove the ergodicity of the semi-group associated to the operator $\tilde{\boldsymbol{S}}=(1/2)\times\sum_{i,j=1}^2\partial_i(\tilde{\boldsymbol{a}}_{ij}(x_1,x_2)\partial_j)$ acting on periodic functions of two variables ($\tilde{\boldsymbol{a}}=\tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\sigma}}$). Basically, this holds true because of the ellipticity of $\tilde{\boldsymbol{a}}$ on the cell $\mathcal{C}=]0,2\pi[^2$.

Here is a precise argument. We denote by X the diffusion process with generator $\tilde{\boldsymbol{S}}$. It is sufficient to establish that, for a given starting point $x\in\mathcal{C}$ and a given Borel subset $B\subset\mathcal{C}$, with $\lambda_{\mathrm{Leb}}(B)>0$ (λ_{Leb} denotes the Lebesgue measure), $\mathbb{P}_x(X_t\in B)>0$. For such a set and $n\in\mathbb{N}^*$, we put $B_n=\mathcal{C}_n\cap B$, with $\mathcal{C}_n=]1/n,2\pi-1/n[\times]1/n,2\pi-1/n[$. We can choose n large enough to ensure $\lambda_{\mathrm{Leb}}(B_n)>0$ and $x\in\mathcal{C}_n$. Moreover, we can modify the coefficient $\tilde{\boldsymbol{\sigma}}$ out of $\mathcal{C}_n$ so that the modified coefficient $\tilde{\boldsymbol{\sigma}}^n$ is periodic and uniformly elliptic on the torus. We then denote by X^n the diffusion process with generator $\tilde{\boldsymbol{S}}^n=(1/2)\sum_{i,j=1}^2\partial_i(\tilde{\boldsymbol{a}}^n_{ij}(x_1,x_2)\partial_j)$. We have

$$\mathbb{P}_x(X_t\in B)\geq\mathbb{P}_x(X_t\in B_n;\forall s<t,X_s\in\mathcal{C}_n)=\mathbb{P}_x(X_t^n\in B_n;\forall s<t,X_s^n\in\mathcal{C}_n).$$

This latter quantity is strictly positive by the uniform ellipticity of $\tilde{\boldsymbol{a}}^n$ (see [23]).

We prove that the effective diffusion coefficient is null. We can consider the column vector $V = (1,0)^*$ and the sequence of 2π-periodic functions $(\boldsymbol{\varphi}_n)_n$ defined, for $(x_1, x_2) \in \mathcal{C}$, by $\boldsymbol{\varphi}_n(x_1, x_2) = 1$ if $x_1 \in [n^{-1}, 2\pi - n^{-1}]$ and $\boldsymbol{\varphi}_n(x_1, x_2) = 1 - n\pi$ otherwise.

The vector $(\boldsymbol{\varphi}_n(x_1, x_2), 0)^*$ corresponds to the gradient of the function $F_n(x_1, x_2) = \int_0^{x_1} \boldsymbol{\varphi}_n(u, x_2)\, du$ and thus belongs to $\mathbb{D}$. Keeping the notations of Subsection 3.2, (9) yields

$$\begin{aligned}
\langle V, \alpha V \rangle &\leq \frac{1}{4\pi^2} \int_{\mathcal{C}} |\boldsymbol{\sigma}(V - DF_n)|^2 dx_1 dx_2 \\
&\leq \frac{3}{4\pi} \int_0^{2\pi} (1 - \boldsymbol{\varphi}_n)^2 (1 - \cos(x_1))^2\, dx_1 = \frac{3n^2\pi}{2} \int_0^{n^{-1}} (1 - \cos(x_1))^2\, dx_1.
\end{aligned}$$

An easy calculation proves that the latter quantity converges to zero as n goes to infinity so that α degenerates along the x_1-axis. The same argument holds for the x_2-axis. Therefore, the matrix α is null.

From a probabilistic point of view, the diffusion process X cannot leave the cell from which it starts. As a consequence, the rescaled process $(X_t^\varepsilon = \varepsilon X_{t/\varepsilon^2})_{t \geq 0}$, which corresponds to the rescaled operator $\tilde{\boldsymbol{S}}^\varepsilon = (1/2) \sum_{i,j=1}^2 \partial_i (\tilde{\boldsymbol{a}}_{ij}(x_1/\varepsilon, x_2/\varepsilon) \partial_j)$, cannot leave the cell of diameter $2\pi\varepsilon$ from which it starts. The limit process is thus constant and the effective diffusion matrix is zero. $\square$

4.2 A Random Chessboard Structure Example

We now set out an example in the stationary framework (and in the two-dimensional setting). We fix a parameter $0 < p < 1$ and we consider, as random medium, the set $\Omega = [0,1]^2 \times \{0,1\}^{\mathbb{Z}^2}$, equipped with the product σ-field and with the following product measure: the two first marginal distributions are uniform distributions on $[0,1]$ and the other ones are Bernoulli distributions of parameter p. We can check that the transformations

$$\forall y \in \mathbb{R}^2,\ \forall \omega = (u, (a_k)_{k \in \mathbb{Z}^2}) \in \Omega,\ \tau_y \omega = (u + y - \lfloor u + y \rfloor, (a_{\lfloor k + u + y \rfloor})_{k \in \mathbb{Z}^2})$$

fit Definition 2.1, where, for $y \in \mathbb{R}^2$, $\lfloor y \rfloor$ stands for the vector whose coordinates are the integer parts of the coordinates of y. Roughly speaking, we are drawing a chessboard on $\mathbb{R}^2$ whose origin is randomly chosen over $[0,1]^2$. We are then coloring each square either in black with probability p or in white with probability $1 - p$.

We tackle the construction of $\tilde{\boldsymbol{\sigma}}$. We define $\mathcal{D}$ as the 2×2 matrix with $\mathcal{D}_{1,1} = 1$ and $\mathcal{D}_{i,j} = 0$ for $i \neq 1$ or $j \neq 1$. Then, we put

$$\forall \omega = (u, (a_k)_{k \in \mathbb{Z}^2}) \in \Omega,\ \hat{\boldsymbol{\sigma}}(\omega) = a_0 I_2 + (1 - a_0)\mathcal{D}.$$

An easy calculation proves that for each $\omega = (u, (a_k)_{k\in\mathbb{Z}^2}) \in \Omega$ and each $y \in \mathbb{R}^2$, $\hat{\boldsymbol{\sigma}}(\tau_y\omega) = a_{\lfloor y+u\rfloor} I_2 + (1 - a_{\lfloor y+u\rfloor})\mathcal{D}$: for a fixed environment ω, the matrix $\tilde{\sigma}(\omega, y)$ is equal to I_2 on black squares and to $\mathcal{D}$ on white ones. We now regularize $\hat{\boldsymbol{\sigma}}$: we choose a smooth density ϱ on $\mathbb{R}^2$ with a very small support and we put $\tilde{\boldsymbol{\sigma}} = \hat{\boldsymbol{\sigma}} \star \varrho$. The ergodicity property for $\boldsymbol{L}^{\tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\sigma}}}$ is very intuitive. Indeed, the matrix $\tilde{\sigma}(\omega, \cdot)$ only degenerates on white squares, and in fact only on a part of each of them (depending on the support of ϱ) and only along the y_2-axis direction: while lying on the degenerating part of a white square, the diffusion associated to $(1/2)\sum_{i,j=1}^{2} \partial_i((\tilde{\sigma}\tilde{\sigma})_{i,j}(y_1, y_2)\partial_j)$ can only move along the y_1-axis direction. Nevertheless, with probability 1, the process encounters a black square sooner or later (since the parameter p belongs to $]0, 1[$): it thus manages to move up and down and hence to reach every given square. Ergodicity follows. Rigorous arguments are however left to the reader. It is plain to prove that the matrix α given by (9) is nondegenerate. $\square$

4.3 Comparison with Existing Literature

We let the reader check that the previous examples do not satisfy a Muckenhoupt condition, as expressed in [4, 8, 17, 18, 19]. Conversely, if a diffusion matrix $\tilde{\boldsymbol{a}}$ satisfies a Muckenhoupt condition, then it is ergodic in the sense of Assumption 2.5 because of [17, Cor. 2.5] and [18, Th. 2.8]. We also emphasize that the Muckenhoupt condition prevents the homogenized diffusion coefficient $\tilde{\boldsymbol{a}}$ from degenerating. Indeed, for a smooth function $\boldsymbol{\varphi}$ defined on Ω and $X \in \mathbb{R}^d$, we have ($\boldsymbol{\lambda}(\omega)$ denotes the smallest eigenvalue of $\tilde{\boldsymbol{a}}(\omega)$) $|X|^2 = \big|\mathbb{E}(X + D\boldsymbol{\varphi})\big|^2 \le \big(\mathbb{E}[\boldsymbol{\lambda}^{-1}]\big)\big(\mathbb{E}\big[(X + D\boldsymbol{\varphi})^*\tilde{\boldsymbol{a}}(X + D\boldsymbol{\varphi}\big]\big)$. Because of the Muckenhoupt condition, $\boldsymbol{\lambda}^{-1}$ is integrable. The nondegeneracy of the effective diffusion coefficient associated to $\tilde{\boldsymbol{a}}$ follows from (9).

5 Preliminary Results for the Proof of Theorem 3.6

5.1 Auxiliary Problems

We now investigate the auxiliary problems and deduce, as a by-product, Proposition 3.4. The solvability of the linear auxiliary problem, related to $\tilde{\boldsymbol{\sigma}}$ is standard, as well as the variational formula (see [21]). Thus, we just focus on the construction of $\boldsymbol{\xi}$.

Approximated Auxiliary Problems. For $\boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathcal{C}$, we set (we extend in an obvious manner the notation $(\cdot,\cdot)_2^\Omega$ to $[L^2(\Omega)]^d$) $(\boldsymbol{\varphi}, \boldsymbol{\psi})_{1,2}^\Omega = -(\boldsymbol{\varphi}, \tilde{\boldsymbol{S}}\boldsymbol{\psi})_2^\Omega = (1/2)\big(\tilde{\boldsymbol{a}}D\boldsymbol{\varphi}, D\boldsymbol{\psi}\big)_2^\Omega$, and the associated seminorm $\|\boldsymbol{\varphi}\|_{1,2}^\Omega = [(\boldsymbol{\varphi}, \boldsymbol{\varphi})_{1,2}^\Omega]^{1/2}$. Then, we can set, for any $\boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathcal{C}$, $\mathcal{E}(\boldsymbol{\varphi}, \boldsymbol{\psi}) = (\boldsymbol{\varphi}, \boldsymbol{\psi})_2^\Omega + (\boldsymbol{\varphi}, \boldsymbol{\psi})_{1,2}^\Omega$. This defines an inner product on $\mathcal{C} \times \mathcal{C}$ and we denote by $\mathbb{H}_1$ the completion of $\mathcal{C}$ for the resulting norm. By the regularity of $\tilde{\boldsymbol{a}}$, $\mathcal{E}$ is closable and $\mathbb{H}_1$ may be seen as a subspace of $L^2(\Omega)$. Equipped with the norm induced by $\mathcal{E}$, $\mathbb{H}_1$ is a Hilbert space.

For any $\boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathcal{C}$, we have $(\boldsymbol{\varphi}, \boldsymbol{\psi})_{1,2}^{\Omega} = (1/2)(\tilde{\boldsymbol{\sigma}} D\boldsymbol{\varphi}, \tilde{\boldsymbol{\sigma}} D\boldsymbol{\psi})_2^{\Omega}$, so that the mapping $\Xi : \mathcal{C} \to \mathbb{D}$, $\boldsymbol{\varphi} \mapsto \tilde{\boldsymbol{\sigma}} D\boldsymbol{\varphi}$ can be extended to the whole space $\mathbb{H}_1$. For each function $\boldsymbol{\varphi} \in \mathbb{H}_1$, we denote $\Xi(\boldsymbol{\varphi})$ by $\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\varphi}$: this represents in a way the gradient of the function $\boldsymbol{\varphi}$ along the direction $\tilde{\boldsymbol{\sigma}}$.

For $\lambda > 0$ and $z \in \mathbb{R}^d$, we can consider the approximated corrector equation $\lambda\boldsymbol{\chi}^\lambda - \boldsymbol{div}(\tilde{\boldsymbol{\sigma}}\boldsymbol{A}(\cdot, \tilde{\boldsymbol{\sigma}}z + \nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\chi}^\lambda(\cdot, z))) = 0$, i.e. for all $\boldsymbol{\varphi} \in \mathbb{H}_1$,

$$\lambda\mathbb{E}\big[\boldsymbol{\chi}^\lambda\boldsymbol{\varphi}\big] + \mathbb{E}\big[\langle \boldsymbol{A}(\cdot, \tilde{\boldsymbol{\sigma}}z + \nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\chi}^\lambda(\cdot, z)), \nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\varphi}\rangle\big] = 0. \tag{16}$$

The nonlinear operator $\boldsymbol{\mathcal{A}}^\lambda : \boldsymbol{\psi} \in \mathbb{H}_1 \mapsto \lambda\boldsymbol{\psi} - \boldsymbol{div}(\tilde{\boldsymbol{\sigma}}\boldsymbol{A}(\cdot, \tilde{\boldsymbol{\sigma}}z + \nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\psi})) \in \mathbb{H}_1'$ is strictly monotone and Lipschitz continuous on $\mathbb{H}_1$, so that the equation $\boldsymbol{\mathcal{A}}^\lambda(\boldsymbol{\psi}) = 0$ admits a unique solution, denoted by $\boldsymbol{\chi}^\lambda(\cdot, z)$ (see [24, Th. 26.A]). We let the reader prove

Lemma 5.1. *There exists a constant $C_{5.1}$ such that, for all $\lambda > 0$ and $z \in \mathbb{R}^d$, $\lambda\mathbb{E}[|\boldsymbol{\chi}^\lambda(\cdot, z)|^2] + \mathbb{E}[|\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\chi}^\lambda(\cdot, z)|^2] \leq C_{5.1}(1 + |z|^2)$.*

Convergence and Regularity of the Approximated Correctors.

Proposition 5.2. *For all $z \in \mathbb{R}^d$, the equation* (10) *admits a unique (weak) solution in $\mathbb{D}$. In particular, Proposition 3.4 holds (the proof of* (11) *follows from the proof of Theorem 3.5). Moreover,* $\lim_{\lambda\to 0} \mathbb{E}[\lambda|\boldsymbol{\chi}^\lambda(\cdot, z)|^2 + |\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\chi}^\lambda(\cdot, z) - \boldsymbol{\xi}(\cdot, z)|^2] = 0$.

Proof. Similarly to (16), we seek for a field $\boldsymbol{\xi}(\cdot, z) \in \mathbb{D}$ such that, for all $\boldsymbol{\varphi} \in \mathbb{H}_1$, $\mathbb{E}[\langle \boldsymbol{A}(\cdot, \tilde{\boldsymbol{\sigma}}z + \boldsymbol{\xi}(\cdot, z)), \nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\varphi}\rangle] = 0$.

Considering the nonlinear operator $\boldsymbol{\mathcal{A}} : \boldsymbol{\theta} \in \mathbb{D} \mapsto -\boldsymbol{div}(\tilde{\boldsymbol{\sigma}}\boldsymbol{A}(\cdot, \tilde{\boldsymbol{\sigma}}z + \boldsymbol{\theta})) \in \mathbb{D}'$, we can prove as above that the equation (10) admits a unique solution ($\mathbb{D}$ is a closed subspace of $[L^2(\Omega)]^d$ and is, for this reason, reflexive). Choosing $\boldsymbol{\varphi} = \boldsymbol{\chi}^\lambda(\cdot, z)$, for a given $\lambda > 0$, we obtain $\mathbb{E}[\langle \boldsymbol{A}(\cdot, \tilde{\boldsymbol{\sigma}}z + \boldsymbol{\xi}(\cdot, z)), \nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\chi}^\lambda(\cdot, z)\rangle] = 0$.

Since $\boldsymbol{\xi}(\cdot, z)$ belongs to $\mathbb{D}$, we can find a sequence $(\boldsymbol{\varphi}_n)_{n\geq 1}$ in $\mathcal{C}$, such that $\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\varphi}_n \to \boldsymbol{\xi}(\cdot, z)$ in $[L^2(\Omega)]^d$. In particular, by the (I_d, Λ)-strict monotonicity of $\boldsymbol{A}(\omega, \cdot)$ for each $\omega \in \Omega$ and by Lemma 5.1, there exists a sequence $(\varepsilon_n(\lambda))_{n\geq 1}$, vanishing as $n \to +\infty$, uniformly in λ, such that $\mathbb{E}[\langle \boldsymbol{A}(\cdot, \tilde{\boldsymbol{\sigma}}z + \nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\varphi}_n), \nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\chi}^\lambda(\cdot, z)\rangle] = \varepsilon_n(\lambda)$.

Making the difference with (16) (with $\boldsymbol{\varphi} = \boldsymbol{\chi}^\lambda(\cdot, z)$), we deduce that $\lambda\mathbb{E}[(\boldsymbol{\chi}^\lambda(\cdot, z))^2] + \mathbb{E}[\langle \boldsymbol{A}(\cdot, \tilde{\boldsymbol{\sigma}}z + \nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\chi}^\lambda(\cdot, z)) - \boldsymbol{A}(\cdot, \tilde{\boldsymbol{\sigma}}z + \nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\varphi}_n), \nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\chi}^\lambda(\cdot, z)\rangle] = -\varepsilon_n(\lambda)$, that is,

$$\begin{aligned}
&\lambda\mathbb{E}\big[\big(\boldsymbol{\chi}^\lambda(\cdot, z)\big)^2\big] + \mathbb{E}\big[\langle \boldsymbol{A}(\cdot, \tilde{\boldsymbol{\sigma}}z + \nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\chi}^\lambda(\cdot, z)) - \boldsymbol{A}(\cdot, \tilde{\boldsymbol{\sigma}}z + \nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\varphi}_n), \nabla^{\tilde{\boldsymbol{\sigma}}}\big(\boldsymbol{\chi}^\lambda(\cdot, z) - \boldsymbol{\varphi}_n\big)\rangle\big] \\
&= -\varepsilon_n(\lambda) - \mathbb{E}\big[\langle \boldsymbol{A}(\cdot, \tilde{\boldsymbol{\sigma}}z + \nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\chi}^\lambda(\cdot, z)) - \boldsymbol{A}(\cdot, \tilde{\boldsymbol{\sigma}}z + \nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\varphi}_n), \nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\varphi}_n\rangle\big] \\
&= -\varepsilon_n(\lambda) + \lambda\mathbb{E}\big[\boldsymbol{\chi}^\lambda(\cdot, z)\boldsymbol{\varphi}_n\big] + \mathbb{E}\big[\langle \boldsymbol{A}(\cdot, \tilde{\boldsymbol{\sigma}}z + \nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\varphi}_n), \nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\varphi}_n\rangle\big].
\end{aligned}$$

By Lemma 5.1, $\lambda\boldsymbol{\chi}^\lambda(\cdot,z)\to 0$ in $L^2(\Omega)$ as λ tends to 0. Moreover, $\mathbb{E}[\langle \boldsymbol{A}(\cdot,\tilde{\boldsymbol{\sigma}}z+\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\varphi}_n),\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\varphi}_n\rangle]=\mathbb{E}[\langle \boldsymbol{A}(\cdot,\tilde{\boldsymbol{\sigma}}z+\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\varphi}_n)-\boldsymbol{A}(\cdot,\tilde{\boldsymbol{\sigma}}z+\boldsymbol{\xi}(\cdot,z)),\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\varphi}_n\rangle]\to 0$ as $n\to+\infty$. Hence, we can first fix n large enough and then λ small enough to let the right-hand side in the above expression be small. □

Proposition 5.3. *There exists a constant $C_{5.3}$ such that, for all $h\in\mathbb{R}^d$,*

$$\sup_{\lambda>0,z\in\mathbb{R}^d}\big\{\lambda\mathbb{E}\big[|\boldsymbol{\chi}^\lambda(\cdot,z+h)-\boldsymbol{\chi}^\lambda(\cdot,z)|^2\big]+\mathbb{E}\big[|\nabla^{\tilde{\boldsymbol{\sigma}}}\big(\boldsymbol{\chi}^\lambda(\cdot,z+h)-\boldsymbol{\chi}^\lambda(\cdot,z)\big)|^2\big]\big\}\le C_{5.3}|h|^2.$$

In particular, the convergence in Proposition 5.2 is uniform on compact subsets of $\mathbb{R}^d$.

Proof. Fix $\lambda>0$ and $z,h\in\mathbb{R}^d$ and consider $\boldsymbol{v}=\boldsymbol{\chi}^\lambda(\cdot,z+h)-\boldsymbol{\chi}^\lambda(\cdot,z)$. From (16), we can write for $\boldsymbol{\varphi}\in\mathbb{H}_1$

$$\lambda\mathbb{E}\big[\boldsymbol{v}\boldsymbol{\varphi}\big]+\mathbb{E}\big[\langle \boldsymbol{A}(\cdot,\tilde{\boldsymbol{\sigma}}(z+h)+\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\chi}^\lambda(\cdot,z+h))-\boldsymbol{A}(\cdot,\tilde{\boldsymbol{\sigma}}z+\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\chi}^\lambda(\cdot,z)),\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\varphi}\rangle\big]=0.$$

Choosing $\boldsymbol{\varphi}=\boldsymbol{v}$, we obtain

$$\begin{aligned}&\lambda\mathbb{E}\big[\boldsymbol{v}^2\big]+\mathbb{E}\big[\langle \boldsymbol{A}(\cdot,\tilde{\boldsymbol{\sigma}}(z+h)+\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\chi}^\lambda(\cdot,z+h))-\boldsymbol{A}(\cdot,\tilde{\boldsymbol{\sigma}}z+\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\chi}^\lambda(\cdot,z)),\tilde{\boldsymbol{\sigma}}h+\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{v}\rangle\big]\\&\quad=\mathbb{E}\big[\langle \boldsymbol{A}(\cdot,\tilde{\boldsymbol{\sigma}}(z+h)+\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\chi}^\lambda(\cdot,z+h))-\boldsymbol{A}(\cdot,\tilde{\boldsymbol{\sigma}}z+\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\chi}^\lambda(\cdot,z)),\tilde{\boldsymbol{\sigma}}h\rangle\big].\end{aligned}$$

Since $\boldsymbol{A}(\omega,\cdot)$ is (I_d,Λ)-strictly monotone for each $\omega\in\Omega$, there exist $C,C'\ge 0$ such that

$$\lambda\mathbb{E}\big[\boldsymbol{v}^2\big]+\mathbb{E}\big[|\tilde{\boldsymbol{\sigma}}h+\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{v}|^2\big]\le C\mathbb{E}\big[|\tilde{\boldsymbol{\sigma}}h+\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{v}||\tilde{\boldsymbol{\sigma}}h|\big]\le C'|h|\mathbb{E}\big[|\tilde{\boldsymbol{\sigma}}h+\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{v}|^2\big]^{1/2}. \quad □$$

5.2 Nonlinear Ergodic Theorem

The following result is the key point in our method. Using the ergodic properties of $\tilde{\boldsymbol{S}}$ (see Assumption 2.5), we establish a nonlinear version of the ergodic theorem on the random medium (the term *nonlinear* indicates that the functionals that we investigate depend on the solutions of (1)). As prescribed in [1], the strategy consists in introducing the resolvent equation associated with $\tilde{\boldsymbol{S}}$.

Theorem 5.4. *Let $\boldsymbol{\Psi}\in L^2(\Omega\times[0,T]\times\mathcal{O})$ and $h:[0,T]\times\mathcal{O}\times\mathbb{R}\to\mathbb{R}$ be a measurable function, such that $\sup_{(t,x)\in\mathcal{Q}_T}|h(t,x,0)|<+\infty$ and $y\in\mathbb{R}\mapsto h(t,x,y)$ is Lipschitz continuous, uniformly in $(t,x)\in\mathcal{Q}_T$. Setting $\bar{\Psi}(t,x)=\mathbb{E}[\boldsymbol{\Psi}(\cdot,t,x)]$, we claim,*

$$\text{(17)}\quad \lim_{\varepsilon\to 0}\sup_{0\le t\le T}\mathbb{E}\bigg|\int_0^t\int_{\mathcal{O}}\big[\Psi(x/\varepsilon,s,x)h(s,x,u^\varepsilon(s,x))-\bar{\Psi}(s,x)h(s,x,u^\varepsilon(s,x))\big]dxds\bigg|=0.$$

Proof. We first assume that there exist a function $\boldsymbol{\psi} \in L^\infty(\Omega)$, with $\mathbb{E}[\boldsymbol{\psi}] = 0$, a real $R_0 > 0$ and a smooth function $\varphi : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, satisfying $\varphi(t,x) = 0$ for $\mathrm{dist}(x, \partial\mathcal{O}) \le 1/R_0$, such that $\boldsymbol{\Psi}(\omega,t,x) = \boldsymbol{\psi}(\omega)\varphi(t,x)$. We also consider a smooth function $h : [0,T] \times \mathcal{O} \times \mathbb{R} \to \mathbb{R}$, satisfying $h(t,x,y) = 0$ for $(t,x) \in [0,T] \times \mathbb{R}^d$ and $|y| > R_0$.

We consider the resolvent equation $\lambda \boldsymbol{v}^\lambda - \boldsymbol{div}(\tilde{\boldsymbol{\sigma}} \nabla^{\tilde{\boldsymbol{\sigma}}} \boldsymbol{v}^\lambda) = \boldsymbol{\psi}$. For a test function $\theta \in \mathcal{C}_K^\infty(\mathcal{O})$, we can integrate the resolvent equation against $\boldsymbol{\ell} \star \theta$, for any $\boldsymbol{\ell} \in L^\infty(\Omega)$. We deduce, for a.e. $\omega \in \Omega$,

$$\lambda \int_{\mathcal{O}} v^\lambda(x/\varepsilon)\theta(x)dx + \varepsilon \int_{\mathcal{O}} \langle \nabla^{\tilde{\sigma}} v^\lambda(x/\varepsilon), \tilde{\sigma}(x/\varepsilon)\nabla\theta(x)\rangle dx = \int_{\mathcal{O}} \psi(x/\varepsilon)\theta(x)dx.$$

Up to a regularization argument for $u^\varepsilon(t,.)$, we can choose $\theta(x) = \varphi(t,x)h(t,x,u^\varepsilon(t,x))$ for a.e. $t \in [0,T]$. We deduce that there exists a constant C, depending on $\boldsymbol{\psi}$, φ and h, such that

$$\begin{aligned}
&\sup_{0 \le t \le T} \mathbb{E}\Big[\Big|\int_0^t \int_{\mathcal{O}} \psi(x/\varepsilon)\varphi(s,x)h(s,x,u^\varepsilon(s,x))dxds\Big|\Big] \\
&\quad \le C\Big[\lambda \mathbb{E}[|\boldsymbol{v}^\lambda|] + \varepsilon\big(\mathbb{E}[|\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{v}^\lambda|^2]\big)^{1/2}\Big]\Big[1 + \mathbb{M}^{dt,dx}\big[|\nabla_x^{\tilde{\sigma}(x/\varepsilon)} u^\varepsilon(t,x)|^2\big]\Big].
\end{aligned}$$

By the ergodicity of the operator associated to $\tilde{\boldsymbol{a}}$, we know that $\lambda^2\mathbb{E}[(\boldsymbol{v}^\lambda)^2] \to 0$ as $\lambda \to 0$ (see [14] for a particular case or [21] and references therein for the general case). By Proposition 3.3, we can first choose λ small enough and then ε small enough to let the above right-hand side be small. This completes the proof in this first case.

We know that the linear combinations of functions of the type $(\omega,t,x) \mapsto \boldsymbol{\varphi}(\omega)\eta(t)\times \rho(x)$ with $\boldsymbol{\varphi} \in L^\infty(\Omega)$, $\eta \in C^\infty([0,T])$ and $\rho \in C_c^\infty(\mathcal{O})$ are dense in the space $L^2(\Omega \times [0,T] \times \mathcal{O})$. As a by-product, (17) still holds for $\boldsymbol{\Psi} \in L^2(\Omega \times [0,T] \times \mathcal{O})$ and h as described above. Details are left to the reader.

We now assume that h is just bounded in (t,x) and Lipschitz continuous in y, as prescribed in the statement. We claim

$$\lim_{R \to 0} \limsup_{\varepsilon \to 0} \mathbb{M}^{dt,dx}\big[\mathbf{1}_{\{|u^\varepsilon(t,x)| > R\}} |\Psi(x/\varepsilon,t,x)||u^\varepsilon(t,x)|\big] = 0. \tag{18}$$

By (18), we can assume, without loss of generality, that the support of h is compact. We then complete the proof by approximating h by a sequence of smooth functions $(h_n)_{n\ge1}$, vanishing for large values of y, such that $\int_0^T \int_{\mathcal{O}} \sup_{y\in\mathbb{R}} |h(t,x,y) - h_n(t,x,y)|^2 dtdx \to 0$.

We prove (18). By the Cauchy-Schwarz inequality and Proposition 3.3, it is sufficient to prove $\lim_{R\to+\infty} \limsup_{\varepsilon\to0} \mathbb{M}^{dt,dx}[\Psi^2(x/\varepsilon,t,x)\mathbf{1}_{\{|u^\varepsilon(t,x)|>R\}}] = 0$. For all $R > 0$ and $\varepsilon >$

0, the stationarity property yields $\mathbb{M}^{dt,dx}[\Psi^2(x/\varepsilon,t,x)\mathbf{1}_{\{|u^\varepsilon(t,x)|>R\}}] = \mathbb{M}^{dt,dx}[\Psi^2(\omega,t,x) \times\mathbf{1}_{\{|u^{\varepsilon,\tau_{-x/\varepsilon}\omega}(t,x)|>R\}}]$. By a uniform integrability argument, it is sufficient to investigate the measure of the set $\{|u^{\varepsilon,\tau_{-x/\varepsilon}\omega}(t,x)| > R\}$ for large values of R. Again by the stationarity property and Proposition 3.3, we have $\mathbb{Q}^{dt,dx}[\{|u^{\varepsilon,\tau_{-x/\varepsilon}\omega}(t,x)| > R\}] \le (1/R)\mathbb{M}^{dt,dx}[|u^{\varepsilon,\omega}(t,\, x)|] \le C/R$. □

6 Proof of Theorem 3.6

As explained in Introduction, we use the first order approximation method to establish Theorem 3.6 (see (4)). Unfortunately, because of the degeneracies of $\tilde{a}$, the solution of the limit PDE as well as the solutions of the auxiliary problems are not are regular enough to do it straight. This is the reason why we introduce a specific regularization procedure.

Regularization Procedure. We first introduce regular versions of the PDE (13).

We denote by p a one-dimensional mollifier. For for all $n \ge 1$, we put $p_n(\cdot) = np(n\,\cdot)$ and we denote by η_n a smooth function from $\mathbb{R}^d$ into $[0,1]$, such that $\eta_n(x) = 1$ for $\mathrm{dist}(x,\mathcal{O}^c) \ge 2/n$ ($\mathcal{O}^c$ denotes the complementary of $\mathcal{O}$) and $\eta_n(x) = 0$ for $\mathrm{dist}(x,\mathcal{O}^c) \le 1/n$. We assume that $\|\nabla\eta_n\|_\infty^{\mathcal{O}} \le \gamma n$, for some constant $\gamma > 0$. Denoting by $*$ the convolution product, we set, for $(x,y,z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$, $\bar{F}_n = \{[(\eta_n \otimes 1_1 \otimes 1_d) \times \bar{F}] * p_{n^2}^{\otimes 2d+1}\}(x,y,z)$ with $(\eta_n \otimes 1_1 \otimes 1_d)(x,y,z) = \eta_n(x)$. Then, $\bar{F}_n(x,\cdot,\cdot)$ is bounded at $(0,0)$ by $\gamma' C_{3.5}$, for a constant $\gamma' > 0$, and is $((1,\varsigma),C_{3.5})$-Lipschitz continuous. Using Footnotes 1 and 2, we can prove

$$\forall R > 0,\ \lim_{n\to+\infty}\Big\|\sup_{|y|\le R,|\varsigma z|\le R}|\bar{F}(\cdot,y,z) - \bar{F}_n(\cdot,y,z)|\Big\|_2^{\mathcal{O}} = 0. \tag{19}$$

Similarly, we put, for $\omega \in \Omega$ and $z \in \mathbb{R}^d$, $\boldsymbol{A}_n(\omega,z) = [\boldsymbol{A}(\omega,\tilde{\boldsymbol{\sigma}}(\omega)\cdot + \boldsymbol{\xi}(\omega,\cdot)) * p_n^{\otimes d}](z)$ if defined and 0 if not (for each $z \in \mathbb{R}^d$, the convolution product is defined for a.e. ω), so that $\boldsymbol{div}(\tilde{\boldsymbol{\sigma}}\boldsymbol{A}_n(\cdot,z)) = 0$. We put $\bar{A}_n(z) = \mathbb{E}[\tilde{\boldsymbol{\sigma}}\boldsymbol{A}_n(\cdot,z)]$, so that $\bar{A}_n = \bar{A} * p_n^{\otimes d}$ (see (12)) and $\bar{A}_n$ is $(\varsigma,\gamma' C_{3.5})$-strictly monotone (up to a modification of γ'). By (11),

$$\lim_{n\to+\infty}\sup_{z\in\mathbb{R}^d}\mathbb{E}\big[|\boldsymbol{A}_n(\cdot,z) - \boldsymbol{A}(\cdot,\tilde{\boldsymbol{\sigma}}z + \boldsymbol{\xi}(\cdot,z))|^2\big] = 0. \tag{20}$$

We admit for the moment (the proof is given in Section 7)

Theorem 6.1. *For every $n \ge 1$, we denote by $\bar{u}_n$ the solution of the PDE* (13)*, with $(\bar{A}_n,\bar{F}_n)$ as coefficients. Then,*

$$\bar{u}_n(t,\cdot) \xrightarrow[\mathcal{O}]{L^2} \bar{u}(t,\cdot) \ \text{ unif. in } t \in [0,T], \quad \nabla_x^\varsigma \bar{u}_n \xrightarrow[[0,T]\times\mathcal{O}]{L^2} \nabla_x^\varsigma \bar{u}. \tag{21}$$

The functions $(\bar{u}_n)_{n\geq 1}$ are once continuously differentiable (C.D. in short) in t, but are just twice C.D. in x along the directions of $\mathrm{Im}(\varsigma)$. *The derivatives along* $\mathrm{Im}(\varsigma)$ *are denoted by $\nabla^{\varsigma}_x \bar{u}_n$ and $\nabla^{2,\varsigma}_{x,x}\bar{u}_n$: for $(t,x) \in \mathcal{Q}_T$, $\nabla^{\varsigma}_x \bar{u}_n(t,x)$ is an element of $\mathbb{R}^d$ (we can also prove that $\nabla^{\varsigma}_x \bar{u}_n(t,\cdot)$ is as an element of $H^{\varsigma,1}_0(\mathcal{O})$ for all $t \in [0,T]$) and $\nabla^{2,\varsigma}_{x,x}\bar{u}_n(t,x)$ an element of $\mathbb{R}^{d\times d}$. For $\nu_1, \nu_2 \in \mathbb{R}^d$, they are given by*

$$\langle \nabla^{\varsigma}_x \bar{u}_n(t,x), \nu_1 \rangle = \partial_{\varsigma\nu_1}\bar{u}_n(t,x), \ \langle \nu_1, \nabla^{2,\varsigma}_{x,x}\bar{u}_n(t,x)\nu_2\rangle = \partial^2_{\varsigma\nu_1,\varsigma\nu_2}\bar{u}_n(t,x). \tag{22}$$

For each $n \geq 1$,

$$\sup_{\mathcal{Q}_T}\Big[\frac{|\bar{u}_n(t,x)|}{\mathrm{dist}(x,\partial\mathcal{O})}\Big] + \|\nabla^{\varsigma}_x \bar{u}_n\|^{\mathcal{Q}_T}_{\infty} + \|\partial_t \bar{u}_n, \nabla^{2,\varsigma}_{x,x}\bar{u}_n\|^{\mathcal{Q}_T}_2 < +\infty, \tag{23}$$

so that $\mathrm{div}(\bar{A}_n(\nabla_x \bar{u}_n))$ *exists as an element of $L^2(\mathcal{Q}_T)$* ([3]).

Moreover, for each $n \geq 1$, there exists a sequence $(\hat{u}_{n,m})_{m\geq 1}$ of continuous functions on $[0,T]\times\mathbb{R}^d$, infinitely C.D. in space on $[0,T]\times\mathbb{R}^d$, once C.D. in time on $\bar{\mathcal{Q}}_T$, such that $(\partial_t \hat{u}_{n,m})_{m\geq 1}$ are infinitely C.D. in space on $\mathcal{Q}_T$, and

$$\begin{aligned} &\sup_{m\geq 1}\big[\|m(\hat{u}_{n,m}-\bar{u}_n)\|^{[0,T]\times\mathbb{R}^d}_{\infty}, \|\nabla^{\varsigma}_x \hat{u}_{n,m}\|^{\mathcal{Q}_T}_{\infty}\big] < +\infty, \\ &\nabla^{\varsigma}_x \hat{u}_{n,m} \overset{\text{Pointwise}}{\underset{\mathcal{Q}_T}{\longrightarrow}} \nabla^{\varsigma}_x \bar{u}_n, \quad \big(\partial_t \hat{u}_{n,m}, \nabla^{2,\varsigma}_{x,x}\hat{u}_{n,m}\big) \overset{L^2}{\underset{\mathcal{Q}_T}{\longrightarrow}} \big(\partial_t \bar{u}_n, \nabla^{2,\varsigma}_{x,x}\bar{u}_n\big) \quad \text{as } m\to+\infty. \end{aligned} \tag{24}$$

Truncation. For each $m \geq 1$, the function $\hat{u}_{n,m}$ doesn't vanish on $[0,T]\times\partial\mathcal{O}$. For this reason, we set, for all $(t,x)\in\mathcal{Q}_T$, $\bar{u}_{n,m}(t,x) = \hat{u}_{n,m}(t,x)\eta_m(x)$, so that $\bar{u}_{n,m} \in \mathcal{C}^{1,2}(\bar{\mathcal{Q}}_T)$.

Lemma 6.2. *For each $n \geq 1$, there exists a constant $C_{6.2}(n)$ such that, for all $m \geq 1$, $\|\nabla^{\varsigma}_x \bar{u}_{n,m}\|^{\mathcal{Q}_T}_{\infty} \leq C_{6.2}(n)$. In particular, $(\nabla^{\varsigma}_x \bar{u}_{n,m} - \nabla^{\varsigma}_x \hat{u}_{n,m})_{m\geq 1}$ converges to 0 in $L^2(\mathcal{Q}_T)$.*

Proof. For $(t,x)\in\mathcal{Q}_T$, $\nabla^{\varsigma}_x \bar{u}_{n,m}(t,x) = \eta_m(x)\nabla^{\varsigma}_x \hat{u}_{n,m}(t,x) + \hat{u}_{n,m}(t,x)\varsigma\nabla\eta_m(x)$. There is no difficulty to handle the first term since $\sup_{m\geq 1}\|\nabla^{\varsigma}_x \hat{u}_{n,m}\|^{\mathcal{Q}_T}_{\infty}$ is finite. For the second one, we can proceed as follows. For $(t,x)\in\mathcal{Q}_T$, $\hat{u}_{n,m}(t,x)\varsigma\nabla\eta_m(x) = 0$ if $\mathrm{dist}(x,\partial\mathcal{O}) \geq 2/m$. If $\mathrm{dist}(x,\partial\mathcal{O}) \leq 2/m$, $|\hat{u}_{n,m}(t,x)\varsigma\nabla\eta_m\ (x)| \leq \gamma m|\varsigma||\bar{u}_n(t,x)| + \gamma m|\varsigma||\hat{u}_{n,m}(t,x)-\bar{u}_n(t,x)| \leq C(n)$, by (23) and (24). $\square$

Regularization of the correctors. Similarly, we regularize the paths of the approximated correctors $(\boldsymbol{\chi}^{\lambda}(\cdot,z))_{z\in\mathbb{R}^d}$, $\lambda > 0$. For $n \geq 1$ and $z \in \mathbb{R}^d$, we can find $\boldsymbol{\theta}^{\lambda}_n(\cdot,z) \in \mathcal{C}$ such

[3] Referring to Footnotes 1 and 2, we can write $\bar{A}_n(\cdot) = \varsigma\hat{A}_n(\varsigma\cdot)$ for a regular function $\hat{A}_n : \mathbb{R}^d \to \mathbb{R}$. The quantity $\mathrm{div}(\bar{A}_n(\nabla_x \bar{u}_n))$ may be expressed as $\mathrm{div}(\bar{A}_n(\nabla_x \bar{u}_n)) = \mathrm{div}(\zeta\hat{A}_n(\nabla^{\zeta}_x \bar{u}_n)) = \sum_{i,j=1}^d [\partial(\hat{A}_n)_i/\partial z_j](\nabla^{\zeta}_x \bar{u}_n)(\nabla^{2,\zeta}_{x,x}\bar{u}_n)_{i,j} = \langle \partial_z \hat{A}_n(\nabla^{\zeta}_x \bar{u}_n), \nabla^{2,\zeta}_{x,x}\bar{u}_n\rangle$. The detailed proof is given in Section 7.

that $\mathbb{E}[|\boldsymbol{\chi}^\lambda(\cdot,z)-\boldsymbol{\theta}_n^\lambda(\cdot,z)|^2+|\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\chi}^\lambda(\cdot,z)-\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\theta}_n^\lambda(\cdot,z)|^2]\le 1/n$. We put for all $\omega\in\Omega$ and $z\in\mathbb{R}^d$

$$\boldsymbol{\chi}_n^\lambda(\omega,z)=\int_{\mathbb{R}^d}\boldsymbol{\theta}_n^\lambda(\omega,n^{-1}\lfloor nz'\rfloor)p_n^{\otimes d}(z-z')dz'.$$

We can see that $\boldsymbol{\chi}_n^\lambda(\cdot,z)\in\mathcal{C}$ for all $z\in\mathbb{R}^d$ and that, for all $\omega\in\Omega$ and $z\in\mathbb{R}^d$, $\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\chi}_n^\lambda(\omega,z)=[\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\theta}_n^\lambda(\omega,n^{-1}\lfloor n\cdot\rfloor)*p_n^{\otimes d}](z)$ (in particular, the function $(\omega,z)\in\Omega\times\mathbb{R}^d\mapsto(\boldsymbol{\chi}_n^\lambda(\omega,z),\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\chi}_n^\lambda(\omega,z))$ is jointly measurable). Proposition 5.3 yields

$$\lim_{n\to+\infty}\sup_{\lambda>0,z\in\mathbb{R}^d}\mathbb{E}\big[\lambda|\boldsymbol{\chi}_n^\lambda(\cdot,z)-\boldsymbol{\chi}^\lambda(\cdot,z)|^2+|\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\chi}_n^\lambda(\cdot,z)-\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\chi}^\lambda(\cdot,z)|^2\big]=0. \tag{25}$$

Moreover, by Propositions 5.2 and 5.3, for each $n\ge 1$ and every compact subset $\mathcal{K}\subset\mathbb{R}^d$,

$$\sup_{\lambda>0}\sup_{z\in\mathcal{K}}\mathbb{E}\big[|\nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\chi}_n^\lambda(\cdot,z)|^2\big]<+\infty,\ \lim_{\lambda\to 0}\lambda\sup_{z\in\mathcal{K}}\mathbb{E}\big[|\boldsymbol{\chi}_n^\lambda(\cdot,z)|^2+|\partial_z\boldsymbol{\chi}_n^\lambda(\cdot,z)|^2\big]=0. \tag{26}$$

First Order Approximation.

Notation. In the whole proof, $R^\varepsilon_{n,m}$, $\Gamma^\varepsilon_{n,m}$ and $\Delta^\varepsilon_{n,m}$ denote, in a generic way, terms that satisfy $\lim_{\varepsilon\to0}\sup_{0\le t\le T}|R^\varepsilon_{n,m}(t)|=0$ for all $n,m\ge1$, $\lim_{m\to\infty}\sup_{\varepsilon>0}\sup_{0\le t\le T}|\Gamma^\varepsilon_{n,m}(t)|=0$ for all $n\ge1$, and $\lim_{n\to+\infty}\sup_{m\ge1,\varepsilon>0}\sup_{0\le t\le T}|\Delta^\varepsilon_{n,m}(t)|=0$. Their values may change from line to line. Similarly, we denote, in a generic way, by C constants that do not depend on (n,m,ε). The value of C may vary from line to line.

Definition. For $n,m\ge1$, we consider a sequence $(\rho^\varepsilon_{n,m})_{\varepsilon>0}$ of smooth functions, from $\mathcal{O}$ into $[0,1]$, with compact supports, converging towards 1 in $L^2(\mathcal{O})$ (as ε tends to zero). We set for all $\varepsilon>0$ and $(t,x)\in\mathcal{Q}_T$

$$u^\varepsilon_{n,m}(t,x)=\bar{u}_{n,m}(t,x)+\varepsilon\chi_n^{\varepsilon^2}(x/\varepsilon,\nabla_x\bar{u}_{n,m}(t,x))\rho^\varepsilon_{n,m}(x). \tag{27}$$

Since $\nabla_x\bar{u}_{n,m}$ is a smooth function, we can write

$$\varepsilon^2\mathbb{E}\big[|\chi_n^{\varepsilon^2}(x/\varepsilon,\nabla_x\bar{u}_{n,m}(t,x))|^2\big]\le\sup\big\{\varepsilon^2\mathbb{E}[(\boldsymbol{\chi}_n^{\varepsilon^2}(\cdot,z))^2],\ |z|\le\sup_{(t,x)}|\nabla\bar{u}_{n,m}(t,x)|\big\}. \tag{28}$$

By the Jensen inequality, (26) and Theorem 6.1, there exists a constant $B\ge0$ such that

$$\begin{aligned}&\forall t\in[0,T],\ \mathbb{M}^{dx}\big[|u^\varepsilon_{n,m}(t,x)-\bar{u}_{n,m}(t,x)|^2\big]\le R^\varepsilon_{n,m}(t),\\ &\qquad\mathbb{M}^{dx}\big[|u^\varepsilon_{n,m}(t,x)|^2\big]\le B+R^\varepsilon_{n,m}(t)+\Gamma^\varepsilon_{n,m}(t).\end{aligned} \tag{29}$$

Gradient of $u^\varepsilon_{n,m}$. In (27), we can differentiate the involved terms with respect to x along $\tilde\sigma(x/\varepsilon)$

$$
\begin{aligned}
\nabla_x^{\tilde\sigma(x/\varepsilon)} u^\varepsilon_{n,m}(t,x) &= \tilde\sigma(x/\varepsilon)\nabla_x \bar u_{n,m}(t,x) + \nabla^{\tilde\sigma}\chi_n^{\varepsilon^2}(x/\varepsilon, \nabla_x \bar u_{n,m}(t,x)) \\
&\quad + (\rho^\varepsilon_{n,m}(x)-1)\nabla^{\tilde\sigma}\chi_n^{\varepsilon^2}(x/\varepsilon,\nabla_x\bar u_{n,m}(t,x)) \\
&\quad + \varepsilon\rho^\varepsilon_{n,m}(x)\tilde\sigma(x/\varepsilon)\big[\partial_z\chi_n^{\varepsilon^2}(x/\varepsilon,\nabla_x\bar u_{n,m}(t,x))\nabla^2_{x,x}\bar u_{n,m}(t,x)\big] \\
&\quad + \varepsilon\chi_n^{\varepsilon^2}(x/\varepsilon,\nabla_x\bar u_{n,m}(t,x))\tilde\sigma(x/\varepsilon)\nabla\rho^\varepsilon_{n,m}(x) \\
&= \tilde\sigma(x/\varepsilon)\nabla_x\bar u_{n,m}(t,x) + \nabla^{\tilde\sigma}\chi_n^{\varepsilon^2}(x/\varepsilon,\nabla_x\bar u_{n,m}(t,x)) + \sum_{i=1,2,3} T^\varepsilon_{i,n,m}(t,x).
\end{aligned}
\tag{30}
$$

We wish to prove that $T^\varepsilon_{1,n,m}$, $T^\varepsilon_{2,n,m}$ and $T^\varepsilon_{3,n,m}$ vanish in a suitable sense with ε.

Term $T^\varepsilon_{3,n,m}$. By (26) and (28), we claim

$$
\varepsilon^2\mathbb{M}^{dt,dx}\big[|\chi_n^{\varepsilon^2}(x/\varepsilon,\nabla_x\bar u_{n,m}(t,x))|^2|\nabla\rho^\varepsilon_{n,m}(x)|^2\big] \le R^\varepsilon_{n,m}(T)\big[\|\nabla\rho^\varepsilon_{n,m}\|_2^{\mathcal{O}}\big]^2.
$$

The family $(\rho^\varepsilon_{n,m})_{\varepsilon>0}$ is not bounded in $H^1_0(\mathcal{O})$. However, we can choose, for each fixed $n,m\ge 1$, the family $(\rho^\varepsilon_{n,m})_{\varepsilon>0}$ such that $\mathbb{M}^{dt,dx}[(T^\varepsilon_{3,n,m})^2]$ vanishes as ε tends to zero.

Terms $T^\varepsilon_{1,n,m}$ and $T^\varepsilon_{2,n,m}$. Similarly, we can prove that, for each fixed $n,m\ge 1$, the quantities $\mathbb{M}^{dt,dx}[|T^\varepsilon_{1,n,m}(t,x)|^2]$ and $\mathbb{M}^{dt,dx}[|T^\varepsilon_{2,n,m}(t,x)|^2]$ vanish as ε tends to zero.

Convergence of the correctors in the gradient. By (30) and the above analysis,

$$
\mathbb{M}^{dt,dx}\big[|\nabla_x^{\tilde\sigma(x/\varepsilon)}u^\varepsilon_{n,m}(t,x) - \tilde\sigma(x/\varepsilon)\nabla_x\bar u_{n,m}(t,x) - \nabla^{\tilde\sigma}\chi_n^{\varepsilon^2}(x/\varepsilon,\nabla_x\bar u_{n,m}(t,x))|^2\big] \le R^\varepsilon_{n,m}(T).
$$

By (25), we can approximate $\nabla^{\tilde\sigma}\boldsymbol{\chi}_n^{\varepsilon^2}(\cdot,z)$ by $\nabla^{\tilde\sigma}\boldsymbol{\chi}^{\varepsilon^2}(\cdot,z)$ in $L^2(\Omega)$ (as $n\to+\infty$), uniformly in z and in ε. By Proposition 5.3, we can approximate $\nabla^{\tilde\sigma}\boldsymbol{\chi}^{\varepsilon^2}(\cdot,z)$ by $\boldsymbol{\xi}(\cdot,z)$ in $L^2(\Omega)$ (as $\varepsilon\to 0$), uniformly on compact sets. Since $\nabla_x\bar u_{n,m}$ is smooth, we deduce

$$
\begin{aligned}
&\mathbb{M}^{dt,dx}\big[|\nabla_x^{\tilde\sigma(x/\varepsilon)}u^\varepsilon_{n,m}(t,x) - \tilde\sigma(x/\varepsilon)\nabla_x\bar u_{n,m}(t,x) - \xi(x/\varepsilon,\nabla_x\bar u_{n,m}(t,x))|^2\big] \\
&\quad \le R^\varepsilon_{n,m}(T) + \Delta^\varepsilon_{n,m}(T).
\end{aligned}
\tag{31}
$$

By (11), we can control $\tilde{\boldsymbol{\sigma}}z+\boldsymbol{\xi}(\cdot,z)-\boldsymbol{\xi}(\cdot,0)$ in $L^2(\Omega)$ by $|\varsigma z|$, for any $z\in\mathbb{R}^d$. By Lemma 6.2 and Theorem 6.1, $\|\varsigma\nabla_x\bar u_{n,m}-\nabla^\varsigma_x\bar u_n\|_2^{\mathcal{Q}_T}\to 0$ as $m\to+\infty$ and $\|\nabla^\varsigma_x\bar u_n-\nabla^\varsigma_x\bar u\|_2^{\mathcal{Q}_T}\to 0$ as $n\to+\infty$. We obtain

$$
\mathbb{M}^{dt,dx}\big[|\nabla_x^{\tilde\sigma(x/\varepsilon)}u^\varepsilon_{n,m}(t,x)|^2\big] \le C + R^\varepsilon_{n,m}(T) + \Gamma^\varepsilon_{n,m}(T).
\tag{32}
$$

Time derivative of $\bar u_{n,m}$. Computing the derivative with respect to t, we claim

$$
\mathbb{M}^{dt,dx}\big[|\partial_t u^\varepsilon_{n,m}(t,x) - \partial_t\bar u_{n,m}(t,x)|^2\big] \le R^\varepsilon_{n,m}(T).
\tag{33}
$$

Distance between $u^\varepsilon_{n,m}$ ***and*** u^ε**.** By (1), Proposition 3.3, (29), (31), (32) and (33), for all $t \in [0,T]$,

$$
\begin{aligned}
&\mathbb{M}_t^{ds,dx}\big[\partial_t(u^\varepsilon - u^\varepsilon_{n,m})(s,x)(u^\varepsilon - u^\varepsilon_{n,m})(s,x)\big] \\
&\quad + \mathbb{M}_t^{ds,dx}\big[\langle A(x/\varepsilon, \nabla_x^{\tilde\sigma(x/\varepsilon)} u^\varepsilon(s,x)) \\
&\qquad\qquad - A(x/\varepsilon, \nabla_x^{\tilde\sigma(x/\varepsilon)} u^\varepsilon_{n,m}(s,x)), \nabla_x^{\tilde\sigma(x/\varepsilon)}(u^\varepsilon - u^\varepsilon_{n,m})(s,x)\rangle\big] \\
&= -\mathbb{M}_t^{ds,dx}\big[F(x/\varepsilon, x, u^\varepsilon(s,x), \nabla^{\tilde\sigma(x/\varepsilon)} u^\varepsilon(s,x))(u^\varepsilon - u^\varepsilon_{n,m})(s,x)\big] \\
&\quad - \mathbb{M}_t^{ds,dx}\big[\partial_t \bar u_{n,m}(s,x)(u^\varepsilon - u^\varepsilon_{n,m})(s,x)\big] \\
&\quad - \mathbb{M}_t^{ds,dx}\big[\langle A(x/\varepsilon, \nabla_x^{\tilde\sigma(x/\varepsilon)} \bar u_{n,m}(s,x) + \xi(x/\varepsilon, \nabla_x \bar u_{n,m}(s,x))), \\
&\qquad\qquad \nabla_x^{\tilde\sigma(x/\varepsilon)}(u^\varepsilon - u^\varepsilon_{n,m})(s,x)\rangle\big] + R^\varepsilon_{n,m}(t) + \Gamma^\varepsilon_{n,m}(t) + \Delta^\varepsilon_{n,m}(t) \\
&= -S^\varepsilon_{1,n,m}(t) - S^\varepsilon_{2,n,m}(t) - S^\varepsilon_{3,n,m}(t) + R^\varepsilon_{n,m}(t) + \Gamma^\varepsilon_{n,m}(t) + \Delta^\varepsilon_{n,m}(t).
\end{aligned} \tag{34}
$$

Lemma 6.3 (Term $S^\varepsilon_{2,n,m}$)**.** *For $t \in [0,T]$ (we recall that, for $s \in [0,T]$, the term $\bar F_n(x, \bar u_n(s,x), \nabla_x \bar u_n(s,x))$ is well defined since $\bar F_n(x,\cdot,\cdot)$ is $((1,\varsigma), C_{3.5})$-Lipschitz continuous and that the term $\operatorname{div}(\bar A_n(\nabla_x \bar u_n(s,x)))$ exists as an element of $L^2(\mathcal{Q}_T)$, see Footnotes 1, 2 and 3),*

$$
\begin{aligned}
S^\varepsilon_{2,n,m}(t) &= -\mathbb{M}_t^{ds,dx}\big[\bar F_n(x, \bar u_n(s,x), \nabla_x \bar u_n(s,x))(u^\varepsilon - u^\varepsilon_{n,m})(s,x)\big] \\
&\quad + \mathbb{M}_t^{ds,dx}\big[\operatorname{div}\big(\bar A_n(\nabla_x \bar u_n(s,x))\big)(u^\varepsilon - u^\varepsilon_{n,m})(s,x)\big] + R^\varepsilon_{n,m}(t) + \Gamma^\varepsilon_{n,m}(t).
\end{aligned}
$$

Proof. By (23), we know that $\partial_t \bar u_n \in L^2(\mathcal{Q}_T)$. Hence,

$$
\begin{aligned}
S^\varepsilon_{2,n,m}(t) &= \mathbb{M}_t^{ds,dx}\big[\partial_t \hat u_{n,m}(s,x)(u^\varepsilon - u^\varepsilon_{n,m})(s,x)\eta_m(x)\big] \\
&= \mathbb{M}_t^{ds,dx}\big[\partial_t \bar u_n(s,x)(u^\varepsilon - u^\varepsilon_{n,m})(s,x)\big] \\
&\quad + \mathbb{M}_t^{ds,dx}\big[\partial_t \bar u_n(s,x)(u^\varepsilon - u^\varepsilon_{n,m})(s,x)(\eta_m(x) - 1)\big] \\
&\quad + \mathbb{M}_t^{ds,dx}\big[\big(\partial_t \hat u_{n,m}(s,x) - \partial_t \bar u_n(s,x)\big)(u^\varepsilon - u^\varepsilon_{n,m})(s,x)\eta_m(x)\big].
\end{aligned}
$$

By Proposition 3.3, (24) and (29), we deduce that $S^\varepsilon_{2,n,m}(t) = \mathbb{M}_t^{ds,dx}[\partial_t \bar u_n(s,x)(u^\varepsilon - u^\varepsilon_{n,m})(s,x)] + R^\varepsilon_{n,m}(t) + \Gamma^\varepsilon_{n,m}(t)$. From the PDE (13) (for the regularized coefficients), we complete the proof. □

Lemma 6.4 (Term $S^\varepsilon_{3,n,m}$)**.** *For all $t \in [0,T]$,*

$$
\begin{aligned}
-S^\varepsilon_{3,n,m}(t) &\le (2\Lambda)^{-1}\mathbb{M}_t^{ds,dx}\big[|\nabla_x^{\tilde\sigma(x/\varepsilon)}(u^\varepsilon - u^\varepsilon_{n,m})(s,x)|^2\big] \\
&\quad + \mathbb{M}_t^{ds,dx}\big[\operatorname{div}\big(\bar A_n(\nabla_x \hat u_{n,m}(s,x))\big)(u^\varepsilon - u^\varepsilon_{n,m})(s,x)\big] \\
&\quad + R^\varepsilon_{n,m}(t) + \Gamma^\varepsilon_{n,m}(t) + \Delta^\varepsilon_{n,m}(t).
\end{aligned}
$$

Proof. Since $\hat{u}_{n,m} \in \mathcal{C}^{1,2}(\bar{\mathcal{Q}}_T)$, we can write, for every $t \in [0,T]$,

$$\begin{aligned}
S^\varepsilon_{3,n,m}(t) &= \mathbb{M}^{ds,dx}_t \big[\langle A(x/\varepsilon, \nabla^{\tilde{\sigma}(x/\varepsilon)}_x \hat{u}_{n,m}(s,x) + \xi(x/\varepsilon, \nabla_x \hat{u}_{n,m}(s,x))),\\
&\qquad\qquad \nabla^{\tilde{\sigma}(x/\varepsilon)}_x (u^\varepsilon - u^\varepsilon_{n,m})(s,x)\rangle\big]\\
&\quad + \mathbb{M}^{ds,dx}_t \big[\langle A(x/\varepsilon, \nabla^{\tilde{\sigma}(x/\varepsilon)}_x \bar{u}_{n,m}(s,x) + \xi(x/\varepsilon, \nabla_x \bar{u}_{n,m}(s,x)))\\
&\qquad\qquad - A(x/\varepsilon, \nabla^{\tilde{\sigma}(x/\varepsilon)}_x \hat{u}_{n,m}(s,x) + \xi(x/\varepsilon, \nabla_x \hat{u}_{n,m}(s,x))),\\
&\qquad\qquad \nabla^{\tilde{\sigma}(x/\varepsilon)}_x (u^\varepsilon - u^\varepsilon_{n,m})(s,x)\rangle\big]\\
&= S^\varepsilon_{3,n,m}(1,t) + S^\varepsilon_{3,n,m}(2,t).
\end{aligned} \tag{35}$$

Since A is Lipschitz continuous, we deduce

$$\begin{aligned}
|S^\varepsilon_{3,n,m}(2,t)| &\leq (\Lambda/2)\mathbb{M}^{ds,dx}_t \big[|\nabla^{\tilde{\sigma}(x/\varepsilon)}_x (\bar{u}_{n,m} - \hat{u}_{n,m})(s,x)\\
&\qquad + \xi(x/\varepsilon, \nabla_x \bar{u}_{n,m}(s,x)) - \xi(x/\varepsilon, \nabla_x \hat{u}_{n,m}(s,x))|^2\big]\\
&\quad + (2\Lambda)^{-1}\mathbb{M}^{ds,dx}_t \big[|\nabla^{\tilde{\sigma}(x/\varepsilon)}_x (u^\varepsilon - u^\varepsilon_{n,m})(s,x)|^2\big].
\end{aligned}$$

Using (11) and Lemma 6.2, we deduce that the first term in the above right hand side tends to zero as m tends to $+\infty$, uniformly in ε. We deduce

$$|S^\varepsilon_{3,n,m}(2,t)| \leq (2\Lambda)^{-1}\mathbb{M}^{ds,dx}_t \big[|\nabla^{\tilde{\sigma}(x/\varepsilon)}_x (u^\varepsilon - u^\varepsilon_{n,m})(s,x)|^2\big] + \Gamma^\varepsilon_{n,m}(t). \tag{36}$$

Consider now $S^\varepsilon_{3,n,m}(1,t)$. We claim (the proof is given below)

$$\begin{aligned}
&S^\varepsilon_{3,n,m}(1,t)\\
&= -\mathbb{M}^{ds,dx}_t \big[\langle \tilde{\sigma}(x/\varepsilon)\partial_z A_n(x/\varepsilon, \nabla_x \hat{u}_{n,m}(s,x)), \nabla^2_{x,x}\hat{u}_{n,m}(s,x)\rangle (u^\varepsilon - \hat{u}_{n,m})(s,x)\big]\\
&\quad + R^\varepsilon_{n,m}(t) + \Gamma^\varepsilon_{n,m}(t) + \Delta^\varepsilon_{n,m}(t).
\end{aligned} \tag{37}$$

Up to the proof of (37), we can complete the proof of Lemma 6.4. Indeed, we can apply Theorem 5.4 with $\boldsymbol{\Psi}(\omega,t,x) = \langle \tilde{\boldsymbol{\sigma}}\partial_z \boldsymbol{A}_n(\cdot, \nabla_x \hat{u}_{n,m}(t,x)), \nabla^2_{x,x}\hat{u}_{n,m}(t,x)\rangle$ and $h(t,x,y) = y - \hat{u}_{n,m}(t,x)$. The quantity $\bar{\boldsymbol{\Psi}}(t,x)$ is equal to $\langle \partial_z \bar{A}_n(\nabla_x \hat{u}_{n,m}(t,x)), \nabla^2_{x,x}\hat{u}_{n,m}(t,x)\rangle$, that is $\bar{\boldsymbol{\Psi}}(t,x) = \mathrm{div}(\bar{A}_n(\nabla_x \hat{u}_{n,m}(t,x)))$. By (35), (36) and (37), the proof is over.

We prove (37) right now. By (20), (29), (32) and Proposition 3.3, it is sufficient to prove that for every smooth function $\psi : \mathcal{O} \to \mathbb{R}$ with a compact support, for all $\varepsilon > 0$, $n,m \geq 1$ and $t \in [0,T]$ and for a.e. $\omega \in \Omega$,

$$\begin{aligned}
&\int_{\mathcal{O}} \langle A_n(x/\varepsilon, \nabla_x \hat{u}_{n,m}(t,x)), \tilde{\sigma}(x/\varepsilon)\nabla\psi(x)\rangle dx\\
&\qquad = -\int_{\mathcal{O}} \langle \tilde{\sigma}(x/\varepsilon)\partial_z A_n(x/\varepsilon, \nabla_x \hat{u}_{n,m}(t,x)), \nabla^2_{x,x}\hat{u}_{n,m}(t,x)\rangle \psi(x) dx.
\end{aligned}$$

We denote by $\boldsymbol{I}^\varepsilon_{n,m}(t)$ the left hand side and by $\boldsymbol{J}^\varepsilon_{n,m}(t)$ the right hand side. We wish to prove that $\mathbb{E}[\boldsymbol{\varphi}\boldsymbol{I}^\varepsilon_{n,m}(t)] = \mathbb{E}[\boldsymbol{\varphi}\boldsymbol{J}^\varepsilon_{n,m}(t)]$ for each function $\boldsymbol{\varphi} \in \mathcal{C}$. Using the stationarity of the medium, we have, for all $t \in [0,T]$,

$$\begin{aligned}
\mathbb{M}^{dx}\big[\boldsymbol{\varphi}\boldsymbol{I}^\varepsilon_n(t)\big] &= \int_{\mathcal{O}} \langle \mathbb{E}\big[\boldsymbol{\varphi}(\tau_{-x/\varepsilon}\cdot)\tilde{\boldsymbol{\sigma}}\boldsymbol{A}_n(\cdot,\nabla_x\hat{u}_{n,m}(t,x))\big], \nabla\psi(x)\rangle dx \\
&= \mathbb{M}^{dx}\big[\boldsymbol{\varphi}\boldsymbol{J}^\varepsilon_n(t)\big] + 1/\varepsilon \int_{\mathcal{O}} \mathbb{E}\big[\langle \boldsymbol{A}_n(\cdot,\nabla_x\hat{u}_{n,m}(t,x)), \nabla^{\tilde{\boldsymbol{\sigma}}}\boldsymbol{\varphi}(\tau_{-x/\varepsilon}\cdot)\rangle\big]\psi(x)dx.
\end{aligned}$$

Since $\boldsymbol{div}(\tilde{\boldsymbol{\sigma}}\boldsymbol{A}_n(\cdot,z)) = 0$ for all $z \in \mathbb{R}^d$, the last term above vanishes. $\square$

End of the proof. From (34), Lemmas 6.3 and 6.4, we obtain

$$\begin{aligned}
&\mathbb{M}^{dx}\big[|(u^\varepsilon - u^\varepsilon_{n,m})(t,\cdot)|^2\big] + (2\Lambda)^{-1}\mathbb{M}^{ds,dx}_t\big[|(\nabla^{\tilde{\sigma}(\cdot/\varepsilon)}u^\varepsilon - \nabla^{\tilde{\sigma}(\cdot/\varepsilon)}u^\varepsilon_{n,m})(s,x)|^2\big] \\
&\le -\mathbb{M}^{ds,dx}_t\big[F(x/\varepsilon,x,u^\varepsilon(s,x),\nabla^{\tilde{\sigma}(x/\varepsilon)}u^\varepsilon(s,x))(u^\varepsilon - u^\varepsilon_{n,m})(s,x)\big] \\
&\quad + \mathbb{M}^{ds,dx}_t\big[\bar{F}_n(x,\bar{u}_n(s,x),\nabla_x\bar{u}_n(s,x))(u^\varepsilon - u^\varepsilon_{n,m})(s,x)\big] \\
&\quad + \mathbb{M}^{ds,dx}_t\big\{\big[\mathrm{div}\big(\bar{A}_n(\nabla_x\hat{u}_{n,m}(s,x))\big) - \mathrm{div}\big(\bar{A}_n(\nabla_x\bar{u}_n(s,x))\big)\big](u^\varepsilon - u^\varepsilon_{n,m})(s,x)\big\} \\
&\quad + R^\varepsilon_{n,m}(t) + \Gamma^\varepsilon_{n,m}(t) + \Delta^\varepsilon_{n,m}(t) \\
&= U^\varepsilon_{1,n,m}(t) + U^\varepsilon_{2,n,m}(t) + U^\varepsilon_{3,n,m}(t) + R^\varepsilon_{n,m}(t) + \Gamma^\varepsilon_{n,m}(t) + \Delta^\varepsilon_{n,m}(t).
\end{aligned} \tag{38}$$

We first treat $U^\varepsilon_{1,n,m}$. By the Lipschitz continuity of F in (y,z) and by (29) and (31), we deduce, for a constant $C \ge 0$,

$$\begin{aligned}
&U^\varepsilon_{1,n,m}(t) \\
&\le -\mathbb{M}^{ds,dx}_t\big\{\big[F\big(x/\varepsilon,x,\bar{u}_{n,m}(s,x),(\tilde{\sigma}(\cdot/\varepsilon)\nabla_x\bar{u}_{n,m} + \xi(\cdot/\varepsilon,\nabla_x\bar{u}_{n,m}))(s,x)\big) \\
&\qquad\qquad \times (u^\varepsilon - u^\varepsilon_{n,m})(s,x)\big]\big\} \\
&\quad + C\mathbb{M}^{ds,dx}_t\big[|u^\varepsilon - u^\varepsilon_{n,m}|^2(s,x)\big] + (4\Lambda)^{-1}\mathbb{M}^{ds,dx}_t\big[|\nabla^{\tilde{\sigma}(\cdot/\varepsilon)}_x u^\varepsilon - \nabla^{\tilde{\sigma}(\cdot/\varepsilon)}_x u^\varepsilon_{n,m}|^2(s,x)\big] \\
&\quad + R^\varepsilon_{n,m}(t) + \Gamma^\varepsilon_{n,m}(t) + \Delta^\varepsilon_{n,m}(t).
\end{aligned} \tag{39}$$

For $U^\varepsilon_{2,n,m}$, we have, for all $R \ge 0$,

$$\begin{aligned}
&\big|U^\varepsilon_{2,n,m}(t) - \mathbb{M}^{ds,dx}_t\big[\bar{F}(x,\bar{u}_n(s,x),\nabla_x\bar{u}_n(s,x))(u^\varepsilon - u^\varepsilon_{n,m})(s,x)\big]\big| \\
&\le \mathbb{M}^{ds,dx}_t\big[\sup_{|y|\le R,|\varsigma z|\le R}|\bar{F}_n - \bar{F}|(x,y,z)|u^\varepsilon - u^\varepsilon_{n,m}|(s,x)\big] \\
&\quad + C\mathbb{M}^{ds,dx}_t\big[(1+|\bar{u}_n| + |\nabla^\varsigma_x\bar{u}_n|)(s,x)|u^\varepsilon - u^\varepsilon_{n,m}|(s,x)\mathbf{1}_{\{(|\bar{u}_n|+|\nabla^\varsigma_x\bar{u}_n|)(s,x)>R\}}\big].
\end{aligned}$$

By Proposition 3.3, (19), (21) and (29), we deduce that

$$\begin{aligned}
&|U^\varepsilon_{2,n,m}(t) - \mathbb{M}^{ds,dx}_t[\bar{F}\big(x,\bar{u}_n(s,x),\nabla_x\bar{u}_n(s,x)\big)(u^\varepsilon - u^\varepsilon_{n,m})(s,x)]| \\
&\quad \le R^\varepsilon_{n,m}(t) + \Gamma^\varepsilon_{n,m}(t) + \Delta^\varepsilon_{n,m}(t).
\end{aligned} \tag{40}$$

By Proposition 3.3, Theorem 6.1, Footnote 3 and (29), we can find, for each $n \geq 1$, a constant C_n, such that for all $m \geq 1$,

$$
\begin{aligned}
&|U^{\varepsilon}_{3,n,m}(t)|^2 \\
&\leq \big[\big\|\mathrm{div}\big(\bar{A}_n(\nabla_x \hat{u}_{n,m})\big) - \mathrm{div}\big(\bar{A}_n(\nabla_x \bar{u}_n)\big)\big\|_2^{\mathcal{Q}_T}\big]^2 \mathbb{M}_t^{ds,dx}\big[|u^{\varepsilon} - u^{\varepsilon}_{n,m}|^2(s,x)\big] \\
&\leq C_n\big[\big\|\nabla^{2,\varsigma}_{x,x}(\hat{u}_{n,m} - \bar{u}_n)\big\|_2^{\mathcal{Q}_T} + \big\||\nabla^{2,\varsigma}_{x,x}\bar{u}_n||\nabla^{\varsigma}_x(\hat{u}_{n,m} - \bar{u}_n)|\big\|_2^{\mathcal{Q}_T}\big]^2 \\
&\quad \times \mathbb{M}_t^{ds,dx}\big[|u^{\varepsilon} - u^{\varepsilon}_{n,m}|^2(s,x)\big] \\
&\leq R^{\varepsilon}_{n,m}(t) + \Gamma^{\varepsilon}_{n,m}(t).
\end{aligned}
\tag{41}
$$

By (39), (40) and (41), we deduce

$$
\begin{aligned}
&\mathbb{M}^{dx}\big[|(u^{\varepsilon} - u^{\varepsilon}_{n,m})(t,x)|^2\big] + (4\Lambda)^{-1}\mathbb{M}_t^{ds,dx}\big[|(\nabla^{\tilde{\sigma}(\cdot/\varepsilon)}u^{\varepsilon} - \nabla^{\tilde{\sigma}(\cdot/\varepsilon)}u^{\varepsilon}_{n,m})(s,x)|^2\big] \\
&\leq C\mathbb{M}_t^{ds,dx}\big[|(u^{\varepsilon} - u^{\varepsilon}_{n,m})(s,x)|^2\big] \\
&\quad - \mathbb{M}_t^{ds,dx}\big\{\big[F\big(x/\varepsilon, x, \bar{u}_{n,m}(s,x), (\tilde{\sigma}(\cdot/\varepsilon)\nabla_x \bar{u}_{n,m} + \xi(\cdot/\varepsilon, \nabla_x \bar{u}_{n,m}))(s,x)\big) \\
&\qquad\qquad \times (u^{\varepsilon} - u^{\varepsilon}_{n,m})(s,x)\big]\big\} \\
&\quad + \mathbb{M}_t^{ds,dx}\big[\bar{F}(x, \bar{u}_n(s,x), \nabla_x \bar{u}_n(s,x))(u^{\varepsilon} - u^{\varepsilon}_{n,m})(s,x)\big] + R^{\varepsilon}_{n,m}(t) + \Delta^{\varepsilon}_{n,m}(t) + \Gamma^{\varepsilon}_{n,m}(t).
\end{aligned}
$$

We apply Theorem 5.4 to $\boldsymbol{\Psi}(\omega,t,x) = \boldsymbol{F}(\omega, x, \bar{u}_{n,m}(t,x), \tilde{\boldsymbol{\sigma}}(\omega)\nabla_x \bar{u}_{n,m}(t,x) + \boldsymbol{\xi}(\omega, \nabla_x \bar{u}_{n,m}(t,x)))$ and $h(t,x,y) = y - \bar{u}_{n,m}(t,x)$. By the Lipschitz property of $\bar{F}$, (24), (29) and by Lemma 6.2, this makes the sum of the second and third terms disappear. It remains to choose n large enough to let $\Delta^{\varepsilon}_{n,m}$ be small, m large enough to let $\Gamma^{\varepsilon}_{n,m}$ be small too and then ε small enough to treat $R^{\varepsilon}_{n,m}$. The Gronwall lemma permits to conclude. □

7 Analysis of the Limit PDE

7.1 Main Results and Proof of Theorem 6.1

The proof of the following result is left to the reader

Theorem 7.1. *Under Assumptions 2.3, 2.4 and 2.5, we consider a sequence $(\bar{A}_n, \bar{F}_n)_{n\geq 1}$ of coefficients, satisfying the same monotonicity, growth and Lipschitz continuity properties as $\bar{A}$ and $\bar{F}$. We also assume that*

$$
\begin{aligned}
&\lim_{n\to+\infty} \sup\big\{|\alpha^{1/2}\zeta|^{-1}\|\langle \bar{A}_n(\cdot) - \bar{A}(\cdot), \zeta\rangle\|_{\infty}^{\mathbb{R}^d},\ \alpha^{1/2}\zeta \neq 0,\ \zeta \in \mathbb{R}^d\big\} = 0, \\
&\forall R \geq 0,\ \lim_{n\to+\infty}\Big\|\sup_{|y|\leq R, |\varsigma z|\leq R}|\bar{F}_n(\cdot, y, z) - \bar{F}(\cdot, y, z)|\Big\|_2^{\mathcal{O}} = 0.
\end{aligned}
\tag{42}
$$

For all $n \geq 1$, we denote by $\bar{u}_n$ the solution of the limit PDE (13) *associated to* $(\bar{A}_n, \bar{F}_n)$*. Then,*

$$\bar{u}_n(t,\cdot) \xrightarrow[\mathcal{O}]{L^2} \bar{u}(t,\cdot) \text{ unif. in } t \in [0,T], \quad \nabla_x^{\varsigma}\bar{u}_n \xrightarrow[[0,T]\times\mathcal{O}]{L^2} \nabla_x^{\varsigma}\bar{u}. \tag{43}$$

In the sequel, we prove that the solution of the PDE (13) is smooth in the directions of $\mathrm{Im}(\varsigma)$ if the coefficients are smooth and if the source term vanishes in the neighborhood of $\partial\mathcal{O}$. We then apply this result to the family $(\bar{u}_n)_{n\geq 1}$ given in the previous statement.

Theorem 7.2. *If the coefficients $\bar{A}$ and $\bar{F}$ are smooth, i.e. $\mathcal{C}_b^{\infty}$, and if there exists a real $\rho_{7.2} > 0$ such that $\bar{F}(x,0,0) = 0$ for $\mathrm{dist}(x,\partial\mathcal{O}) \leq \rho_{7.2}$, then the unique solution $\bar{u}$ of the PDE* (13) *belongs to the spaces $\mathcal{C}(\bar{\mathcal{Q}}_T)$ and $\mathcal{C}^{\varsigma,1,2}(\mathcal{Q}_T)$ (i.e. $\partial_t\bar{u}$, $\nabla_x^{\varsigma}\bar{u}$ and $\nabla_{x,x}^{2,\varsigma}\bar{u}$ are continuous on $\mathcal{Q}_T$, see* (22) *for a definition of these notations) and satisfies*

$$\sup_{\mathcal{Q}_T}\Big[\frac{|\bar{u}(t,x)|}{\mathrm{dist}(x,\partial\mathcal{O})}\Big] + \|\nabla_x^{\varsigma}\bar{u}\|_{\infty}^{\mathcal{Q}_T} + \|\partial_t\bar{u}, \nabla_{x,x}^{2,\varsigma}\bar{u}\|_2^{\mathcal{Q}_T} < +\infty. \tag{44}$$

Moreover, the PDE (13) *can be written in a nondivergence form: there exists a smooth mapping Θ from $\mathbb{R}^d$ into $\mathbb{R}^{d\times d}$ such that $\mathrm{div}(\bar{A}(\nabla_x\varphi))(x) = \sum_{i,j=1}^{d}[\Theta_{i,j}(\nabla_x^{\varsigma}\varphi(x))(\nabla_{x,x}^{2,\varsigma}\varphi(x))_{i,j}]$ for all $(t,x) \in \mathcal{Q}_T$ and for all smooth function φ from $\mathbb{R}^d$ into $\mathbb{R}$ and this relationship still holds for $\bar{u}$.*

We will also prove the following approximation result.

Theorem 7.3. *Under the assumptions of Theorem 7.2, we can find a sequence $(\check{u}_m)_{m\geq 1}$ of continuous functions on $[0,T]\times\mathbb{R}^d$, vanishing outside a compact subset of $[0,T]\times\mathbb{R}^d$, infinitely continuously differentiable (C.D. in short) in space on $[0,T]\times\mathbb{R}^d$, once C.D. in time on $\bar{\mathcal{Q}}_T$, such that $(\partial_t\check{u}_m)_{m\geq 1}$ are infinitely C.D. in space on $\mathcal{Q}_T$, $\sup_{m\geq 1}\|\nabla_x^{\varsigma}\check{u}_m\|_{\infty}^{\mathcal{Q}_T}$ is finite, and*

$$\check{u}_m \xrightarrow[[0,T]\times\mathbb{R}^d]{\text{Unif.}} \bar{u}, \quad \nabla_x^{\varsigma}\check{u}_m \xrightarrow[\mathcal{Q}_T]{\text{Pointwise}} \nabla_x^{\varsigma}\bar{u}, \quad \big(\partial_t\check{u}_m, \nabla_{x,x}^{2,\varsigma}\check{u}_m\big) \xrightarrow[\mathcal{Q}_T]{L^2} \big(\partial_t\bar{u}, \nabla_{x,x}^{2,\varsigma}\bar{u}\big). \tag{45}$$

Corollary 7.4. *Theorem 6.1 holds.*

Proof. The convergence of $(\bar{u}_n)_{n\geq 1}$ towards $\bar{u}$ follows from Theorem 7.1 (using the $(\varsigma, C_{3.5})$-monotonicity of $\bar{A}$ and using (19), the reader can check (42)). The regularity of each $\bar{u}_n$, $n \geq 1$, is given by Theorem 7.2. The construction of the smooth approximations of $\bar{u}_n$, for each $n \geq 1$, follows from Theorem 7.3 (set $\hat{u}_{n,m} = \check{u}_m$ with $(\bar{A}_n, \bar{F}_n)$ as underlying coefficients). □

7.2 Strategy for the Proof of Theorems 7.2 & 7.3

The proof relies on a change of coordinates along the eigenvectors of the matrix α. Loosely speaking, in the new coordinates, the PDE (13) may be expressed as a system of nondegenerate PDEs defined on a smaller space than $\mathbb{R}^d$, the system being parameterized by the kernel of the matrix α. We are then able to exploit the standard theory for nondegenerate equations.

In what follows, the assumptions of Theorem 7.2 are in force.

Proposition 7.5. *Let $\bar{u}$ denote the solution of* (13) *(in the sense of Theorem 3.5), r denote the rank of α, $(\lambda_i)_{1\le i\le r}$ stand for the non-zero eigenvalues of α, M be a $d\times d$ orthogonal matrix such that $M\alpha M^* = J_r$, J_r being the $d\times d$ diagonal matrix with $(\lambda_1,\dots,\lambda_r,0,\dots)$ as diagonal and $\bar{v}$ be the function given by $\bar{v}(t,x)=\bar{u}(t,M^*x)$ for all $(t,x)\in\hat{\mathcal{Q}}_T=[0,T]\times\hat{\mathcal{O}}$, $\hat{\mathcal{O}}=M\mathcal{O}$. Let I_r denote the $d\times d$ diagonal matrix of rank r with $(1,\dots,1,0,\dots)$ as diagonal (i.e. with r "ones" and $d-r$ "zeros" on the diagonal). If $\bar{v}$ belongs to the spaces $\mathcal{C}(\mathrm{Closure}(\hat{\mathcal{Q}}_T))$ and $\mathcal{C}^{I_r,1,2}(\hat{\mathcal{Q}}_T)$ and satisfies* (44) *with respect to $\partial\hat{\mathcal{O}}$ (instead of $\partial\mathcal{O}$), to $\hat{\mathcal{Q}}_T$ (instead of $\mathcal{Q}_T$) and to I_r (instead of ς), then Theorem 7.2 holds.*

Similarly, if we can find a sequence $(\check{v}_m)_{m\ge1}$ of continuous functions on $[0,T]\times\mathbb{R}^d$, vanishing outside a compact subset of $[0,T]\times\mathbb{R}^d$, infinitely C.D. in space on $[0,T]\times\mathbb{R}^d$, once C.D. in time on $\mathrm{Closure}(\hat{\mathcal{Q}}_T)$, such that $(\partial_t\check{v}_m)_{n\ge1}$ are infinitely C.D. differentiable in space on $\hat{\mathcal{Q}}_T$, $\sup_{m\ge1}\|\nabla_x^{\hat{\varsigma}}\check{v}_m\|_\infty^{\hat{\mathcal{Q}}_T}$ is finite and (45) *holds with respect to $\hat{\mathcal{Q}}_T$ (instead of $\mathcal{Q}_T$), I_r (instead of ς) and $(\check{v}_m,\bar{v})$ (instead of $(\check{u}_m,\bar{u})$), then Theorem 7.3 holds.*

Proof. For $(t,x)\in\mathcal{Q}_T$, $z\in\mathbb{R}^d$ and λ small enough, we can write $\bar{u}(t,x+\lambda\varsigma z)=\bar{v}(t,Mx+\lambda M\varsigma z)$. Since ς is the symmetric square root of α, we have $M\varsigma M^*=J_r^{1/2}$, so that $M\varsigma z=J_r^{1/2}Mz$ belongs to $E_r=\mathrm{Vect}(e_1,\dots,e_r)$. We deduce that $\bar{u}$ is differentiable with respect to x along $\mathrm{Im}(\varsigma)$. The same argument holds for the second order derivatives. Moreover, $\nabla_x^{\varsigma}\bar{u}(t,x)=\varsigma M^*\nabla_x^{I_r}\bar{v}(t,Mx)$ and $\nabla_{x,x}^{2,\varsigma}\bar{u}(t,x)=\varsigma M^*\nabla_{x,x}^{2,I_r}\bar{v}(t,Mx)M\varsigma$.

Now, we can give a sense to $\mathrm{div}(\bar{A}(\nabla_x\bar{u}))$. According to Footnotes 1 and 2, we can find an $(I_d,C_{3.5})$-strictly monotone function $\hat{A}$ such that $\bar{A}(z)=\varsigma\hat{A}(\varsigma z)$ for all $z\in\mathbb{R}^d$. Since $\bar{A}$ is smooth, we can assume that $\hat{A}$ is also smooth. Hence, $\bar{A}(\nabla_x\bar{u}(t,x))$ may be expressed in a more rigorous way as $\varsigma\hat{A}(\varsigma M^*\nabla_x^{I_r}\bar{v}(t,Mx))$. For every test function $\psi\in\mathcal{C}_K^\infty(\mathcal{O})$ and every $t\in[0,T]$

$$\int_{\mathcal{O}}\langle\varsigma\hat{A}(\varsigma M^*\nabla_x^{I_r}\bar{v}(t,Mx)),\nabla_x\psi(x)\rangle dx=\int_{\hat{\mathcal{O}}}\langle M\varsigma\hat{A}(\varsigma M^*\nabla_x^{I_r}\bar{v}(t,x)),\nabla_x(\psi(M^*x))\rangle dx.$$

Since $(M\varsigma)_{i,j}=0$ for $r\le i\le d$, $1\le j\le d$, we can compute $\mathrm{div}(M\varsigma\hat{A}(\varsigma M^*\nabla_x^{I_r}\bar{v}(t,x)))$. It is equal to $\sum_{i,j=1}^r[\partial\hat{A}_i/\partial z_j](\varsigma M^*\nabla_x^{I_r}\bar{v}(t,x))(\varsigma M^*\nabla_{x,x}^{2,I_r}\bar{v}(t,x)M\varsigma)_{i,j}=\sum_{i,j=1}^r[\partial\hat{A}_i/\partial z_j]$

$(\nabla_x^\varsigma \bar{u}(t, M^*x))(\nabla_{x,x}^{2,\varsigma} \bar{u}(t, M^*x))_{i,j}$. We can easily complete the proof of Theorem 7.2. The proof of Theorem 7.3 is similar. $\square$

We characterize the function $\bar{v}$ as follows

Proposition 7.6. *Let $\hat{A}$ and $\hat{F}$ be smooth functions from $\mathbb{R}^d$ into $\mathbb{R}^d$ and from $\hat{\mathcal{O}}\times\mathbb{R}\times\mathbb{R}^d$ into $\mathbb{R}$ such that $(\bar{A}(z), \bar{F}(x, y, z)) = (\varsigma\hat{A}(\varsigma z), \hat{F}(Mx, y, \varsigma z))$ for $(x, y, z) \in \mathcal{O}\times\mathbb{R}\times\mathbb{R}^d$ (see Footnotes 1 and 2 for their construction) and let $\hat{\varsigma} = \varsigma M^*$ and $R = (\mathbf{1}_{\{i=j\}})_{1\le i\le d, 1\le j\le r}$ (R is a $d\times r$ matrix). For $w \in \mathbb{R}^{d-r}$, we denote by $\hat{\mathcal{O}}^w$ the open set $\{X \in \mathbb{R}^r,\ (X, w) \in \hat{\mathcal{O}}\}$ and by $\mathcal{I}$ the set $\{w \in \mathbb{R}^{d-r},\ \hat{\mathcal{O}}^w \neq \emptyset\}$. For each $w \in \mathcal{I}$, we consider the PDE*

$$
\begin{aligned}
\frac{\partial U}{\partial t}(t, X) - \sum_{i=1}^{r} \frac{\partial}{\partial X_i}\big(R^*\hat{\varsigma}^*\hat{A}(\hat{\varsigma}R\nabla_X U(t, X))\big)_i& \\
+ \hat{F}\big(X, w, U(t, X), \hat{\varsigma}R\nabla_X U(t, X)\big) = 0,&
\end{aligned}
\tag{46}
$$

$(t, X) \in]0, T]\times\hat{\mathcal{O}}^w$, with the boundary condition $U(t, X) = 0$ for $t = 0$ and for $X \in \partial\hat{\mathcal{O}}^w$.

Assume that for every $w \in \mathcal{I}$, we can find a strong solution $U(\cdot, \cdot; w)$ to the PDE (46) in the space $\mathcal{C}(\mathrm{Closure}(\hat{\mathcal{Q}}_T^w)) \cap \mathcal{C}^{1,2}(\hat{\mathcal{Q}}_T^w)$ (with $\hat{\mathcal{Q}}_T^w = [0, T] \times \hat{\mathcal{O}}^w$), such that

$$
\sup_{w\in\mathcal{I}}\big[\|\nabla_X U(\cdot, \cdot; w)\|_\infty^{\hat{\mathcal{Q}}_T^w} + \|\partial_t U(\cdot, \cdot; w)\|_2^{\hat{\mathcal{Q}}_T^w} + \|\nabla_{X,X}^2 U(\cdot, \cdot; w)\|_2^{\hat{\mathcal{Q}}_T^w}\big] < +\infty.
$$

Assume also that the function V, given by $V(t, (X, w)) = U(t, X; w)$ for $w \in \mathcal{I}$ and $(t, X) \in \hat{\mathcal{Q}}_T^w$ and $V(t, (X, w)) = 0$ elsewhere, is continuous, vanishes on $\{0\} \times \hat{\mathcal{O}} \cup [0, T] \times \partial\hat{\mathcal{O}}$ and satisfies $\sup\{|V(t, (X, w))|/\mathrm{dist}((X, w), \partial\hat{\mathcal{O}}),\ (t, (X, w)) \in \hat{\mathcal{Q}}_T\} < +\infty$. Then, the functions $\bar{v}$ and V coincide.

Proof. Since $\bar{u}$ satisfies the PDE (13), we can prove by a change of variable that the function $\bar{v}$ belongs to $L^2(]0, T[, H_0^{\hat{\varsigma},1}(\hat{\mathcal{O}}))$ and that $\partial_t\bar{v}$ belongs to $L^2(]0, T[, H^{\hat{\varsigma},-1}(\hat{\mathcal{O}}))$. Moreover, $\bar{v}$ satisfies the PDE

$$
\frac{\partial \bar{v}}{\partial t}(t, x) - \mathrm{div}\big(\hat{\varsigma}^*\hat{A}(\nabla_x^{\hat{\varsigma}}\bar{v}(t, x))\big) + \hat{F}\big(x, \bar{v}(t, x), \nabla_x^{\hat{\varsigma}}\bar{v}(t, x)\big) = 0,
\tag{47}
$$

with $\bar{v}(0, \cdot) = 0$. This equation is uniquely solvable, so that we can complete the proof of Proposition 7.6 by proving that the function V satisfies (47) on $\hat{\mathcal{Q}}_T$.

To prove that V satisfies the PDE (47), we consider, for all $n \ge 1$, a smooth function $\eta_n : \mathbb{R}^d \to [0, 1]$ such that $\eta_n(x) = 1$ if $\mathrm{dist}(x, \hat{\mathcal{O}}^c) \ge 2/n$ and $\eta_n(x) = 0$ if $\mathrm{dist}(x, \hat{\mathcal{O}}^c) \le 1/n$ ($\hat{\mathcal{O}}^c$ denotes the complementary of $\hat{\mathcal{O}}$). For a d-dimensional mollifier $p^{(d)}$, we can set, for all $n \ge 1$, $p_n^{(d)} = n^d p^{(d)}(n\cdot)$ and $V_n(t, \cdot) = (V(t, \cdot)\eta_n) * p_{n^2}^{(d)}$

for all $t \in [0,T]$. Since V is continuously differentiable on $\hat{\mathcal{Q}}_T$ with respect to the r first coordinates and since the kernel of $\hat{\varsigma}$ corresponds to the $d-r$ last coordinates, we claim that $\hat{\varsigma}\nabla_x V_n(t,\cdot) = (\eta_n \hat{\varsigma} R \nabla_X V(t,\cdot)) * p^{(d)}_{n^2} + (V(t,\cdot)\hat{\varsigma}\nabla_x \eta_n) * p^{(d)}_{n^2}$. Since $\sup\{|V(t,(X,w))|/\mathrm{dist}((X,w),\partial\hat{\mathcal{O}}),\ (t,(X,w)) \in \hat{\mathcal{Q}}_T\} < +\infty$, it is plain to deduce that $(V_n(t,\cdot))_{n\geq 1}$ converges towards $V(t,\cdot)$ for every $t \in [0,T]$ with respect to the norm $N^{\hat{\varsigma}}$ and that $V \in L^\infty([0,T], H_0^{\hat{\varsigma},1}(\hat{\mathcal{O}}))$. Of course, $\nabla_x^{\hat{\varsigma}} V = \hat{\varsigma} R \nabla_X V$. Moreover, we know that $\partial_t V \in L^2(]0,T[\times\hat{\mathcal{O}})$ and thus to $L^2(]0,T[, H^{\hat{\varsigma},-1}(\hat{\mathcal{O}}))$. Now, for every smooth function $\varphi \in \mathcal{C}_K^\infty(\hat{\mathcal{O}})$ and for every $0 \leq t \leq T$, we deduce from (46) and from the equivalence $((X,w) \in \hat{\mathcal{O}} \Leftrightarrow w \in \mathcal{I}$ and $X \in \hat{\mathcal{O}}^w)$

$$
\begin{aligned}
&\int_0^t \int_{\hat{\mathcal{O}}} \langle \hat{A}(\hat{\varsigma} R \nabla_X V(s,x)), \hat{\varsigma}\nabla\varphi(x)\rangle dx ds \\
&= \int_{\mathcal{I}} \left[\int_0^t \int_{\hat{\mathcal{O}}^w} \langle \hat{A}(\hat{\varsigma} R \nabla_X U(s,X;w)), \hat{\varsigma} R \nabla_X \varphi(X,w)\rangle dX ds \right] dw \\
&= -\int_{\hat{\mathcal{O}}} V(t,x)\varphi(x) dx + \int_{\hat{\mathcal{O}}} V(0,x)\varphi(x) dx - \int_0^t \int_{\hat{\mathcal{O}}} \hat{F}\big(x, V(s,x), \nabla_x^{\hat{\varsigma}} V(s,x)\big)\varphi(x) ds dx.
\end{aligned}
$$

This completes the proof. $\square$

Theorems 7.2 and 7.3 follow from Proposition 7.5 and 7.6 and the following

Theorem 7.7. *For $w \in \mathcal{I}$, the PDE (46) admits a unique strong solution $U(\cdot,\cdot;w)$ satisfying the conditions required in the statement of Proposition 7.6. Moreover, the function V, given by $V(t,(X,w)) = U(t,X;w)$ for $w \in \mathcal{I}$ and $(t,X) \in \hat{\mathcal{Q}}_T^w$ and $V(t,(X,w)) = 0$ elsewhere, as well as the function $\bar{v}$, given by $\bar{v}(t,x) = \bar{u}(t,M^*x)$, $(t,x) \in \hat{\mathcal{Q}}_T$, coincide and fulfill the conditions exhibited in the statement of Proposition 7.5.*

7.3 Proof of Theorem 7.7

For $w \in \mathcal{I}$, the PDE (46) may be expressed under the following (nondivergence) form

$$
\begin{aligned}
&\frac{\partial U}{\partial t}(t,X) - \sum_{i,j=1}^r \big(\hat{\varsigma}^* \partial_z \hat{A}(\hat{\varsigma} R \nabla_X U(t,X))\hat{\varsigma}\big)_{i,j} \frac{\partial^2 U}{\partial X_i \partial X_j}(t,X) \\
&\qquad + \hat{F}\big(X, w, U(t,X), \hat{\varsigma} R \nabla_X U(t,X)\big) = 0,\ (t,X) \in]0,T] \times \hat{\mathcal{O}}^w,
\end{aligned}
\tag{48}
$$

with the boundary condition $U(t,X) = 0$ for $t = 0$ or $X \in \partial\hat{\mathcal{O}}^w$. We claim

Lemma 7.8. *For $w \in \mathcal{I}$, the PDE (48) admits a unique strong solution $U(\cdot,\cdot;w)$, that is Hölder continuous, with a bounded gradient, on the closure of $\hat{\mathcal{Q}}_T^w$, and whose partial* *derivatives of order one in t and of order two in x are Hölder continuous on every compact subset of $\hat{\mathcal{Q}}_T^w$.*

Proof. We aim at applying [10, Th. 6.2, Ch. 5]. Since the matrices $((\varsigma^* \partial_z \hat{A}(z)\varsigma)_{1\le i,j\le r})_{z\in\mathbb{R}^d}$ are uniformly nondegenerate, the coefficients of (46) satisfy the required assumptions. The whole point is to verify that the section $\hat{\mathcal{O}}^w$ is smooth.

Generally speaking, the sections of a smooth domain may not be smooth. Because of the convexity of the domain, this is true in our setting (see Lemma 7.9 below). This completes the proof. □

Lemma 7.9. *The set $\mathcal{I}$ is a bounded open convex subset of $\mathbb{R}^{d-r}$. Moreover, for every $w \in \mathcal{I}$, $\hat{\mathcal{O}}^w$ is an open convex subset of $\mathbb{R}^r$.*

For $w \in \mathcal{I}$ and $(X, w) \in \partial\hat{\mathcal{O}}$, we can find a non-empty ball B, of center (X, w), and a smooth mapping φ from B to $\mathbb{R}$, with a non-zero gradient, such that, for all $(Y, z) \in B$, $(Y, z) \in \hat{\mathcal{O}}$ (resp. $\partial\hat{\mathcal{O}}$) iff $\varphi(Y, z) < 0$ (resp. $\varphi(Y, z) = 0$). Then, $\nabla_X \varphi(X, w) \neq 0$ and we can find a non-empty ball B^w, of center X, such that, for all $Y \in B^w$, $Y \in \hat{\mathcal{O}}^w$ (resp. $\partial\hat{\mathcal{O}}^w$) iff $\varphi(Y, w) < 0$ (resp. $\varphi(Y, w) = 0$). In particular, for $w \in \mathcal{I}$, $(X, w) \in \partial\hat{\mathcal{O}}$ iff $X \in \partial\hat{\mathcal{O}}^w$ and the boundary of $\hat{\mathcal{O}}^w$ has the same regularity as the boundary of $\mathcal{O}$.

Proof. Left to the reader. □

Lemma 7.10. *There exist two constants $0 < \alpha_{7.10} \le 1$ and $C_{7.10}$ such that, for all $w \in \mathcal{I}$, $\|U(\cdot,\cdot;w)\|_\infty^{\hat{\mathcal{Q}}_T^w} \le C_{7.10}$, and for all $(t, X), (t', X') \in \hat{\mathcal{Q}}_T^w$, $|U(t', X'; w) - U(t, X; w)| \le C_{7.10}[|t' - t|^{\alpha_{7.10}/2} + |X' - X|^{\alpha_{7.10}}]$.*

Proof. By the nondivergence form (48) and the maximum principle, we can establish the uniform boundedness of the family $(\|U(\cdot,\cdot;w)\|_\infty^{\hat{\mathcal{Q}}_T^w})_{w\in\mathcal{I}}$. To obtain the uniform Hölder continuity of the mappings $(U(\cdot,\cdot;w))_{w\in\mathcal{I}}$, we can apply [10, Th. 1.1, Ch. 5] on each $\hat{\mathcal{Q}}_T^w$, $w \in \mathcal{I}$. Since the sets $(\hat{\mathcal{Q}}_T^w)_{w\in\mathcal{I}}$ are all convex, the required Condition (A) (see [10, p. 9]) is fulfilled for each of them with $a_0 = 1$ and $\theta_0 = 1/2$. □

Lemma 7.11. *There exists $C_{7.11} \ge 0$ such that, for all $w \in \mathcal{I}$, $\|\nabla_X U(\cdot,\cdot;w)\|_\infty^{\hat{\mathcal{Q}}_T^w} \le C_{7.11}$. Moreover, there exists $\rho_{7.11} > 0$ such that $U(t, X; w) = 0$ for all $(t, X) \in [0, T] \times \hat{\mathcal{O}}^w$ if $\mathrm{meas}(\hat{\mathcal{O}}^w) < \rho_{7.11}$ ($\mathrm{meas}(\hat{\mathcal{O}}^w)$ stands for the measure of $\hat{\mathcal{O}}^w$) or if $\mathrm{dist}(w, \partial\mathcal{I}) < \rho_{7.11}$.*

Proof. We know that there exists a real $\rho > 0$ such that $\bar{F}(x, 0, 0) = 0$ for $x \in \mathcal{O}$ and $\mathrm{dist}(x, \partial\mathcal{O}) < \rho$, i.e. $\hat{F}(X, w, 0, 0) = 0$ for $(X, w) \in \hat{\mathcal{O}}$ and $\mathrm{dist}((X, w), \partial\hat{\mathcal{O}}) < \rho$.

We choose $w \in \mathcal{I}$. If $\mathrm{meas}(\hat{\mathcal{O}}^w) < \rho^r$, then for every $X \in \hat{\mathcal{O}}^w$, there exists a point $Y \notin \hat{\mathcal{O}}^w$ such that $\sup_{1\le i\le r} |Y_i - X_i| < \rho$. In particular, for every $X \in \hat{\mathcal{O}}^w$,

$\mathrm{dist}(X, \partial\hat{\mathcal{O}}^w) < \rho$, so that $\mathrm{dist}((X,w), \partial\hat{\mathcal{O}}) < \rho$. Finally, $\hat{F}(X,w,0,0) = 0$ for all $X \in \hat{\mathcal{O}}^w$. It is then clear that $U(t,X;w) = 0$ for all $(t,X) \in [0,T] \times \hat{\mathcal{O}}^w$.

If $\mathrm{dist}(w, \partial\mathcal{I}) < \rho$, then there exists $z \notin \mathcal{I}$ such that $|w - z| < \rho$. For every $X \in \hat{\mathcal{O}}^w$, $|(X,w) - (X,z)| < \rho$ and $(X,z) \notin \hat{\mathcal{O}}$ since $\hat{\mathcal{O}}^z = \emptyset$. Hence, $\mathrm{dist}((X,w), \partial\hat{\mathcal{O}}) < \rho$ so that $\hat{F}(X,w,0,0) = 0$. We conclude as in the previous paragraph.

If $\mathrm{meas}(\hat{\mathcal{O}}^w) > \rho^r$, we first estimate the gradient of $U(\cdot,\cdot;w)$ on the boundary of $\partial\hat{\mathcal{O}}^w$. We can apply the classical barrier techniques for convex domain, see e.g. [6, Cor. 14.3]. We deduce that there exists a constant $\Gamma \geq 0$, not depending on w, such that $\sup\{|\nabla_X U(t,X;w)|, \, (t,X) \in [0,T] \times \partial\hat{\mathcal{O}}^w\} \leq \Gamma$.

We now estimate the gradient inside $\hat{\mathcal{O}}^w$. We aim at applying [10, Th. 4.1, Ch. 5]. To this end, we define the normalized domain $\mathcal{U}^w = (\mathrm{meas}(\hat{\mathcal{O}}^w))^{-1/r}\hat{\mathcal{O}}^w$ (so that the measure of $\mathcal{U}^w$ is equal to one) as well as $\Upsilon(t,Z;w) = U(t, (\mathrm{meas}(\hat{\mathcal{O}}^w))^{1/r}Z;w)$ for all $(t,Z) \in [0,T] \times \mathcal{U}^w$. Then, Υ satisfies a PDE of the same form as (46) (up to rescaling factors that are controlled from above and from below). Since the measure of $\mathcal{U}^w$ is equal to one, we can apply [10, Th. 4.1, Ch. 5]: the gradient of Υ can be estimated in terms of the constant Γ (see the previous paragraph), the regularity of the coefficients (which is independent of the parameter w) and the regularity of the boundary of $\mathcal{U}^w$ (the so-called Condition (A) in [10, p. 9]). Since $\mathcal{U}^w$ is convex, we can choose $(a_0, \theta_0) = (1, 1/2)$ in Condition (A), so that the gradient can be estimated independently of w. □

Lemma 7.12. *There exists a constant $C_{7.12} \geq 0$ such that, for all $t \in [0,T]$ and $(X,w) \in \hat{\mathcal{O}}$, $|U(t,X;w)| \leq C_{7.12}\mathrm{dist}((X,w), \partial\hat{\mathcal{O}})$.*

Proof. Since the function U is bounded, we establish the statement for (X,w) close to the boundary of $\hat{\mathcal{O}}$. To this end, we use a covering argument.

For every $x_0 \in \partial\hat{\mathcal{O}}$, we denote by n_0 the outward normal vector to $\hat{\mathcal{O}}$ at x_0. If one of the r first coordinates of n_0 is non-zero, we can find a non-empty ball B of center x_0 and a constant $C > 0$ such that, for $(X,w) \in \hat{\mathcal{O}} \cap B$, $\mathrm{dist}(X, \partial\hat{\mathcal{O}}^w) \leq C\mathrm{dist}((X,w), \partial\hat{\mathcal{O}})$. We deduce that $|U(t,X;w)| \leq C \times C_{7.11}\mathrm{dist}((X,w), \partial\hat{\mathcal{O}})$, for all $(t,(X,w)) \in [0,T] \times (\hat{\mathcal{O}} \cap B)$.

If the r first coordinates of n_0 are all equal to zero, the tangent space to $\hat{\mathcal{O}}$ at x_0 is orthogonal to the kernel of $\hat{\varsigma}$. By convexity, we know that $\hat{\mathcal{O}}$ is either above or below the tangent space. In particular, we can find a unitary vector ν, with $\nu_1 = \cdots = \nu_r = 0$, and a non-empty ball B of center x_0 and a real $\varepsilon > 0$ such that, for all $(X,w) \in \hat{\mathcal{O}} \cap B$, $\hat{\mathcal{O}}^{w+\varepsilon\nu} = \emptyset$. We deduce that $\mathrm{dist}(w, \partial\mathcal{I}) \leq \varepsilon$. For $\varepsilon < \rho_{7.11}$, we obtain $U(t,X;w) = 0$ for all $(X,w) \in \hat{\mathcal{O}} \cap B$.

Using a covering argument, we complete the proof. □

Lemma 7.13. *There exists a constant $C_{7.13} \geq 0$ such that, for all $w \in \mathcal{I}$, $\|\partial_t U(\cdot,\cdot;w)\|_2^{\hat{\mathcal{Q}}_T^w} + \|\nabla^2_{X,X} U(\cdot,\cdot;w)\|_2^{\hat{\mathcal{Q}}_T^w} \leq C_{7.13}$.*

Proof. *In the whole proof, all the balls we consider are constructed with respect to the underlying* L^∞ *norm.* We fix $w \in \mathcal{I}$ and we first estimate $\nabla^2_{X,X} U(\cdot, \cdot; w)$ near the boundary of $\partial\hat{\mathcal{O}}^w$. We choose to this end $X^0 \in \partial\hat{\mathcal{O}}^w$. By Lemma 7.9, we can find a real $\varepsilon > 0$ and a mapping $\varphi : B_r(X^0, 2\varepsilon) \times B_{d-r}(w, 2\varepsilon) \to \mathbb{R}$, with a non-zero gradient with respect to the first coordinates (i.e. $\nabla_X \varphi$ is not zero), such that

$$\forall z \in B_{d-r}(w, 2\varepsilon),\ \forall X \in B_r(X^0, 2\varepsilon),\ X \in \hat{\mathcal{O}}^z (\text{resp. } \partial\hat{\mathcal{O}}^z) \Leftrightarrow \varphi(X, z) < 0 \ (\text{resp.} = 0).$$

Modifying the order of the coordinates as well as ε if necessary, we can assume without loss of generality that the rth coordinate of the gradient $\nabla_X \varphi$ doesn't vanish on $B_r(X^0, 2\varepsilon) \times B_{d-r}(w, 2\varepsilon)$. In particular, $\inf_{|y - X_r^0| \le \varepsilon} \varphi(X_1^0, \dots, X_{r-1}^0, y, w) < 0$ and $\sup_{|y - X_r^0| \le \varepsilon} \varphi(X_1^0, \dots, X_{r-1}^0, y, w) > 0$.

By continuity of φ, we can find $0 < \varepsilon' < \varepsilon$ such that

$$\rho_- = \sup\Big\{ \inf_{|X_r - X_r^0| \le \varepsilon} \varphi(X_1, \dots, X_r, z),\ |z - w| \le \varepsilon',\ |X_j - X_j^0| \le \varepsilon',\ 1 \le j \le r-1 \Big\} < 0,$$

$$\rho_+ = \inf\Big\{ \sup_{|X_r - X_r^0| \le \varepsilon} \varphi(X_1, \dots, X_r, z),\ |z - w| \le \varepsilon',\ |X_j - X_j^0| \le \varepsilon',\ 1 \le j \le r-1 \Big\} > 0.$$

Now, we consider, for all $z \in B_{d-r}(w, 2\varepsilon)$, the change of variable $\Psi(\cdot, z) : (X_1, \dots, X_r) \in B(X^0, 2\varepsilon) \mapsto (Y_1, \dots, Y_r) = (X_1 - X_1^0, \dots, X_{r-1} - X_{r-1}^0, \varphi(X_1, \dots, X_r))$. By definition of ρ_- and ρ_+, we can check that, for all $z \in \mathbb{R}^{d-r}$ such that $|z - w| \le \varepsilon'$, the cylinder $]-\varepsilon', \varepsilon'[^{r-1} \times]\rho_-, 0[$ is included in $\Psi(B(X^0, 2\varepsilon) \cap \hat{\mathcal{O}}^z, z)$.

For $|z - w| \le \varepsilon'$, we can write the PDE (46) in the new coordinates $(Y_1, \dots, Y_r)$. For $t \in]0, T]$ and $Y \in]-\varepsilon', \varepsilon'[^{r-1} \times]\rho_-, 0[$, we set $\Upsilon(t, Y; z) = U(t, \Psi^{-1}(Y); z)$, so that $\Upsilon(\cdot, \cdot; z)$ satisfies on $]0, T] \times]-\varepsilon', \varepsilon'[^{r-1} \times]\rho_-, 0[$ (with the notation $(D\Psi)_{i,j} = \partial\Psi_i / \partial X_j$)

$$\begin{aligned}
\frac{\partial \Upsilon}{\partial t}(t, Y) &- \sum_{i=1}^r \frac{\partial}{\partial Y_i} \big[D\Psi(\Psi^{-1}(Y, z), z) R^* \hat{\varsigma}^* \hat{A}\big(\hat{\varsigma} R D\Psi^*(\Psi^{-1}(Y, z), z) \nabla_Y \Upsilon(t, Y) \big) \big]_i \\
&+ \sum_{i=1}^r \Big[\frac{\partial}{\partial Y_i} \big[D\Psi(\Psi^{-1}(Y, z), z) \big] R^* \hat{\varsigma}^* \hat{A}\big(\hat{\varsigma} R D\Psi^*(\Psi^{-1}(Y, z), z) \nabla_Y \Upsilon(t, Y) \big) \Big]_i \\
&+ \hat{F}\big(\Psi^{-1}(Y, z), z, \Upsilon(t, Y), \hat{\varsigma} R D\Psi^*(\Psi^{-1}(Y, z), z) \nabla_Y \Upsilon(t, Y) \big) = 0,
\end{aligned}$$

with the boundary condition $\Upsilon(t, Y; z) = 0$ for $Y_r = 0$. It is crucial to note that all these PDEs are defined on the same domain, with the same boundary condition on the hyperplane $Y_r = 0$. The underlying coefficients are regular with respect to the variable Y, uniformly in $z \in B_r(w, \varepsilon')$. The underlying diffusion matrices are also elliptic, uniformly with respect to Y and z.

We now aim at applying [10, Th. 4.1, Ch. 5]. The difficulty is that the values of the function $\Upsilon(\cdot,\cdot;z)$, for $z \in B_r(w,\varepsilon')$, are unknown on the boundary of $]-\varepsilon',\varepsilon'[^{r-1}\times]-\rho_-,0[$, except on $]-\varepsilon',\varepsilon'[^{r-1}\times\{0\}$. Referring to the proof of [10, Th. 4.1, Ch. 5] (see in particular [10, p. 441 & 442]), this is not a real problem: we can draw a regular open domain $\mathcal{U}$ inside $]-\varepsilon',\varepsilon'[^{r-1}\times]\rho_-,0[$ and containing $]-\varepsilon'/2,\varepsilon'/2[^{r-1}\times]\rho_-/2,0[$. On $\mathcal{U}$, we have $\|\nabla^2_{Y,Y}\Upsilon(\cdot,\cdot;z)\|_2^{[0,T]\times\mathcal{U}} \leq C$ for all $z \in B_{d-r}(w,\varepsilon')$ and for a constant C independent of z. Using the diffeomorphisms $(\Psi(\cdot,z))_{|z-w|\leq\varepsilon'}$, we can pull back these estimates to $(\Psi^{-1}(\mathcal{U},z))_{|z-w|\leq\varepsilon'}$. All the underlying Jacobian determinants are uniformly controlled with respect to z, so that (up to a modification of C) $\|\nabla^2_{X,X}U(\cdot,\cdot;z)\|_2^{[0,T]\times\Psi^{-1}(\mathcal{U},z)} \leq C$ for all $z \in B_{d-r}(w,\varepsilon')$.

Consider now $\mathcal{V} =]-\varepsilon'/2,\varepsilon'/2[^{r-1}\times]\rho_-/2,\rho_+/2[$. It is clear that $\Psi^{-1}(\mathcal{U},z) \supset \Psi^{-1}(\mathcal{V},z) \cap \hat{\mathcal{O}}^z$ for $|z-w| \leq \varepsilon'$. Moreover, the set $\cup_{|z-w|<\varepsilon'}\Psi^{-1}(\mathcal{V},z)\times\{z\}$ is clearly open and contains (X^0,w). Here is the result that we have proved: for all $w \in \mathcal{I}$ and $X^0 \in \partial\hat{\mathcal{O}}^w$, there exist $\eta > 0$ and $C \geq 0$ such that $\|\nabla^2_{X,X}U(\cdot,\cdot;z)\|_2^{[0,T]\times(\hat{\mathcal{O}}^z\cap B_r(X^0,\eta))} \leq C$ for all $z \in B_{d-r}(w,\eta)$.

Consider now $\mathcal{J} = \{w \in \mathcal{I},\ \mathrm{dist}(w,\partial\mathcal{I}) \geq \rho_{7.11}\}$. It is a compact subset of $\mathbb{R}^{d-r}$. By means of Lemma 7.9, we can prove that $\cup_{w\in\mathcal{J}}\partial\hat{\mathcal{O}}^w \times \{w\}$ is a compact subset of $\mathbb{R}^d$. In particular, we can cover it by open sets of the previous form: we can find N points $(X^i,w^i)_{1\leq i\leq N}$, $w^i \in \mathcal{J}$ and $X^i \in \partial\hat{\mathcal{O}}^{w^i}$, as well as N constants $(C_i)_{1\leq i\leq N}$ and N reals $(\eta_i)_{1\leq i\leq N}$, such that

$$\bigcup_{w\in\mathcal{J}}\partial\hat{\mathcal{O}}^w \times \{w\} \subset \bigcup_{i=1}^N B_r(X^i,\eta_i)\times B_{d-r}(w^i,\eta_i), \tag{49}$$

$$\forall 1 \leq i \leq N,\ \forall z \in B_{d-r}(w^i,\eta_i),\ \|\nabla^2_{X,X}U(\cdot,\cdot;z)\|_2^{[0,T]\times(\hat{\mathcal{O}}^z\cap B_r(X^i,\eta_i))} \leq C_i. \tag{50}$$

From (49), we can find a real $\delta > 0$ such that $\forall w \in \mathcal{J},\ \forall X \in \hat{\mathcal{O}}^w,\ \mathrm{dist}(X,\partial\hat{\mathcal{O}}^w) < \delta \Rightarrow (X,w) \in \cup_{i=1}^N B_r(X^i,\eta_i)\times B_{d-r}(w^i,\eta_i)$, so that we can find a constant C' such that

$$\forall w \in \mathcal{J},\ \int_0^T \int_{\{X\in\hat{\mathcal{O}}^w,\ \mathrm{dist}(X,\partial\hat{\mathcal{O}}^w)<\delta\}} |\nabla^2_{X,X}U(t,X;w)|^2 dtdX \leq C'. \tag{51}$$

Now, again for $w \in \mathcal{J}$, we can apply the interior estimates given in [10, (4.7), Ch. 5] on the set $\{X \in \hat{\mathcal{O}}^w,\ \mathrm{dist}(X,\partial\hat{\mathcal{O}}^w) \geq \delta\}$. We deduce (up to a modification of C') that

$$\int_0^T \int_{\{X\in\hat{\mathcal{O}}^w,\ \mathrm{dist}(X,\partial\hat{\mathcal{O}}^w)\geq\delta\}} |\nabla^2_{X,X}U(t,X;w)|^2 dtdX \leq C'. \tag{52}$$

Gathering (51) and (52), we complete the proof for $w \in \mathcal{J}$ (the estimate for $\partial_t U(\cdot,\cdot;w)$ follows from the nondivergence form (48)). If $\mathrm{dist}(w,\partial\mathcal{I}) < \rho_{7.11}$, $U(\cdot,\cdot;w)$ is zero. $\square$

Lemma 7.14. *We set $V(t,(X,w)) = U(t,X;w)$ for $w \in \mathcal{I}$ and $(t,X) \in \hat{\mathcal{Q}}_T^w$ and $V(t,(X,w)) = 0$ elsewhere. The functions $(t,(X,w)) \in [0,T] \times \mathbb{R}^d \mapsto V(t,(X,w))$ and $(t,(X,w)) \in \hat{\mathcal{Q}}_T \mapsto [\partial_t V, \nabla_X V, \nabla^2_{X,X} V](t,(X,w))$ are continuous.*

Proof. For $w \in \mathcal{I}$, we extend the function $U(\cdot,\cdot;w)$ to $[0,T]\times\mathbb{R}^r$ by setting $U(t,X;w) = 0$ if $(t,X) \notin \hat{\mathcal{Q}}_T^w$. By Lemma 7.10, the functions $(U(\cdot,\cdot;w))_{w\in\mathbb{R}^r}$ are equicontinuous.

Now, we consider $w \in \mathcal{I}$ and a sequence $(w_n)_{n\geq 1}$ converging towards w. Since $\mathcal{I}$ is open, we can assume that $(w_n)_{n\geq 1}$ is included in $\mathcal{I}$. Using the equicontinuity property, we can also assume that the $(U(\cdot,\cdot;w_n))_{n\geq 1}$ uniformly converges towards a continuous function U_∞. We first prove that this function is equal to zero on $[0,T] \times \partial\hat{\mathcal{O}}^w$. We choose to this end $X \in \partial\hat{\mathcal{O}}^w$. By Lemma 7.9, (X,w) belongs to $\partial\hat{\mathcal{O}}$. Expressing the boundary in local coordinates, we can find, for all $n \geq 1$, a point $X_n \in \mathbb{R}^r \setminus \hat{\mathcal{O}}^{w_n}$ such that $X_n \to X$. Since $U(t,X_n;w_n) = 0$ for $t \in [0,T]$ and $n \geq 1$, we have $U_\infty(t,X) = 0$.

We can use interior estimates (see [10, Ths. 5.1 & 5.4, Ch. 5]) for the derivative in time and for the second order derivatives in space to prove that the limit function U_∞ satisfies the PDE (46) on $\hat{\mathcal{O}}^w$. By Lemma 7.8, this proves that $U_\infty = U(\cdot,\cdot;w)$.

Now, we consider $w \notin \mathcal{I}$ and a sequence $(w_n)_{n\geq 1}$ converging towards w. If $w \notin \bar{\mathcal{I}}$, it is obvious that $U(\cdot,\cdot;w_n) = 0$ for n large enough so that $(U(\cdot,\cdot;w_n))_{n\geq 1}$ uniformly converges towards $U(\cdot,\cdot;w)$. If $w \in \partial\mathcal{I}$, then $U(\cdot,\cdot;w_n) = 0$ for n large enough by Lemma 7.12 and the sequence $(U(\cdot,\cdot;w_n))_{n\geq 1}$ uniformly converges towards $U(\cdot,\cdot;w)$.

For the continuity of the derivatives, we can proceed as above: on $\mathcal{I}$, we use the interior estimates given in [10, Ths. 5.1 & 5.4, Ch. 5]; on $\mathcal{I}^c$, the result is obvious. $\square$

Lemma 7.15. *There exists a sequence $(V_n)_{n\geq 1}$ of continuous functions on $[0,T] \times \mathbb{R}^d$, with compact supports, infinitely C.D. in space on $[0,T] \times \mathbb{R}^d$, once C.D. in time on* Closure$(\hat{\mathcal{Q}}_T)$*, such that $(\partial_t V_n)_{n\geq 1}$ are infinitely C.D. in space on $\hat{\mathcal{Q}}_T$, $\sup_{n\geq 1} \|\nabla_X V_n\|_\infty^{\hat{\mathcal{Q}}_T} < +\infty$ (∇_X and $\nabla^2_{X,X}$ denote the derivatives along the r first coordinates of the space), and $V_n \underset{[0,T]\times\mathbb{R}^d}{\overset{\text{Unif.}}{\longrightarrow}} V$, $\nabla_X V_n \underset{\hat{\mathcal{Q}}_T}{\overset{\text{Pointwise}}{\longrightarrow}} \nabla_X V$, $\partial_t V_n \underset{\hat{\mathcal{Q}}_T}{\overset{L^2}{\longrightarrow}} \partial_t V$ and $\nabla^2_{X,X} V_n \underset{\hat{\mathcal{Q}}_T}{\overset{L^2}{\longrightarrow}} \nabla^2_{X,X} V$.*

Proof. For a d-dimensional mollifier $p^{(d)}$, we set, for all $n \geq 1$, $p_n^{(d)}(\cdot) = n^d p^{(d)}(n\,\cdot)$. For a given $x_0 \in \hat{\mathcal{O}}$, there exists $N \in \mathbb{N}^*$ such that $B_d(x_0, 1/N) \subset \hat{\mathcal{O}}$. We let the reader check that the following sequence fits all the required conditions:

$$\forall n \geq 1,\ (t,x) \in [0,T]\times\mathbb{R}^d,\ V_n(t,x) = \int_{\mathbb{R}^d} V\big(t, \frac{1}{n+1}x_0 + \frac{n}{n+1}(x-y)\big) p^{(d)}_{(N+1)n}(y)dy. \quad \square$$

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